INTRODUCTION

TO THE

Differential Calculus.

BY

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PREFACE.

THIS Pamphlet embraces the subjects and principles, which, in the form of notes or lectures, have been given to the Classes of the United States Military Academy during the past several years, in order to assist them in understanding the Calculus.

To the Officers of the U. S. Army, who have taught the subject with me, I am greatly indebted for many of the methods and demonstrations here presented.

I am also under additional obligations to Lieuts. EDGERTON, GIBSON, and ALEXANDER, for valuable assistance in preparing the sheets for the printer.

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Greek Alphabet.

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ ς	Sigma
H	η	Eta	T	τ	Tau
Θ	ϑ θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	φ	Phi
K	$\varkappa$	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

DIFFERENTIAL CALCULUS.

CHAPTER I.

CONSTANTS, VARIABLES, AND FUNCTIONS.

1. In the Calculus quantities are divided into two general classes, *constants* and *variables.*

A Constant is a quantity that has, or is supposed to have, a definite fixed value.

A Variable is a quantity that is, or is supposed to be, continually changing in value.

In general, constants are represented by the first letters of the alphabet, and variables by the last; but they should not, therefore, be confused with the known and unknown quantities of Algebra, which, in general, are constants.

The same quantity may sometimes be either a variable or a constant, depending upon the circumstances under which it is considered. Thus, in the equation of a curve, the coördinates of its points are variables; but in the equation of a tangent to the curve, the coördinates of the point of tangency are generally treated as constants. It is, therefore, necessary to determine from the circumstances, or object in view, which quantities are to be regarded as variables, and which as constants, in each discussion.

In general, any or all of the quantities represented by letters in any mathematical expression or equation may have definite values assigned to them, and be regarded as constants; or they may be considered as changing in value, and treated as variables. Thus, in the expression $4\pi r^2$, r is a constant if we suppose it to represent the radius of a particular sphere; but if r is considered as changing in value, it will be a variable. In the first case, $4\pi r^2$ is a constant, and measures the surface of a particular sphere; but when r is variable,

$4\pi r^2$ is also variable, and represents the surface of any sphere no matter how much it may increase or diminish. The formation of a soap bubble illustrates the latter case.

It should not be understood, however, that we may in all cases treat quantities as constants or variables at pleasure without affecting the character of the magnitude represented by the expression or equation. For example, π is generally assumed to represent the ratio of the circumference of any circle to its diameter ~~radius~~, which ratio is invariable. If a different value be assigned to π, the expression $4\pi r^2$ will not measure the surface of a sphere whose radius is r.

In some cases variation in a quantity changes the dimensions of the magnitude represented by the expression or equation; in others it changes the position only; and again it may change the character of the magnitude. Thus, if we suppose R to vary in the equation $(x-\alpha)^2+(y-\beta)^2=R^2$, we shall have a series of circles differing in size; but by changing α or β and not R the position only will be affected.

By changing b^2 within positive limits, the equation $a^2y^2+b^2x^2=a^2b^2$ represents different ellipses, but negative values for b^2 cause the equation to represent hyperbolas. In general, however, constants are supposed to have fixed values in the same expression, unless for a particular discussion it is otherwise stated.

Functions.

2. A quantity is a *function* of another quantity when its value depends upon that of the second quantity. Thus, $4ax$ is a function of 4, a, and x. In general, any mathematical expression which contains a quantity is a function of that quantity. If, however, a quantity disappears from an expression by reduction or simplification the expression is not a function of that quantity. Thus,

$x^2+(c+x)(c-x)$, $\frac{ax}{bx}$, and $\tan x \cot x$, are not functions of x.

3. *A function of a single variable* is one whose value depends upon that of a single variable and varies with it. Thus,

$$\frac{4x^2}{1-x^4}, \qquad \sqrt{r^2x^2+2px}, \qquad \log(a+x), \qquad \sec x,$$

in which x is the only variable, are functions of a single variable.

Any function of a single variable is also a variable, and varies simultaneously with the variable.

4. The relation between a function of a variable and its variable is one of mutual dependence. Any change in the value of one causes a dependent variation in that of the other. Either may, therefore, be regarded as a function of the other; and they are called *inverse functions.* Thus, if x passes from the value 2 to 3, the function $2x^2$ will vary from 8 to 18; and conversely, x will increase from 2 to 3, if $2x^2$ changes from 8 to 18.

In the equation of a curve, the ordinate of any point may be considered as a function of the abscissa, or the abscissa as a function of the ordinate.

The function is considered as dependent, and the variable as independent; for which reason, the latter is called *the independent variable*, or simply *the variable.*

Representing a function of x, as x^3, by y, we have $y=x^3$; solving with respect to x, we have $x=\sqrt[3]{y}$; a form expressing directly x as a function of y.

The difference in form in the following important examples of direct and inverse functions should be observed.

Having, $y=x^n$; then $x=\sqrt[n]{y}$.

$y=a+x$; $x=y-a$.

$y=ax$; $x=\dfrac{y}{a}$.

$y=a^x$; $x=\log_a y$.

5. A *state* of a function *corresponding* to a value or expression for the variable is a *result* obtained by substituting the value or expression for the variable in the function. Thus,

$$-\infty, \quad -16a, \quad -2a, \quad 0, \quad 2a, \quad 16a, \quad \infty,$$

are the states of the function $2ax^3$ corresponding, respectively, to the values or expressions for x,

$$-\infty, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad \infty,$$

and

$$0, \quad \frac{1}{2}, \quad \sqrt{\frac{1}{2}}, \quad 1, \quad 0, \quad -1, \quad 0,$$

are the states of the function $\sin\varphi$ corresponding, respectively, to the expressions or values of φ,

$$0, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi.$$

A function of a variable has an infinite number of states. It may have equal states corresponding to different values of the variable; and it may have two or more states corresponding to the same value of the variable. Thus,

$$5 \text{ and } 1, \quad 7\pm\sqrt{12}, \quad 13 \text{ and } 5, \quad 13\pm\sqrt{24}, \quad 25 \text{ and } 13,$$

are the states of the function $2x+1\pm\sqrt{4x}$, corresponding, respectively, to the values of x,

$$1, \quad 3, \quad 4, \quad 6, \quad 9.$$

Trigonometric functions have equal states for all angles differing by any entire multiple of 2π.

In connection with *any* state of a function corresponding to any value of the variable, it is frequently necessary to consider another state of the function, which results from increasing the value of the variable corresponding to the first state by some convenient arbitrary amount.

In order to distinguish between these two states of the function, the first is designated as a *primitive state*, and the other as *its new* or *second state.*

Any arbitrary amount by which the variable is increased from any assumed value is called *an increment of the variable.* It is generally represented by the letter h, or k, or by Δ written before the variable; as, Δx, read "increment of x".

Let x' represent any particular value of x, and h, or $\Delta x'$, its increment; then will $2ax'^2$ and $2a(x'+h)^2$, or $2a(x'+\Delta x')^2$, represent, respectively, the primitive and new states of the function $2ax^2$, corresponding to x' and its increment h, or $\Delta x'$. The general expression $2a(x+h)^2$ represents the second state of *any* primitive state of the function $2ax^2$, and from it we obtain the second state corresponding to any *particular* primitive state by substituting the proper value of x. The increment of a variable is always assumed as positive.

6. *A function of two or more variables* is one which depends upon two or more variables and varies with each. Thus,

$$x\sin y, \quad xy, \quad x^y, \quad y\log x, \quad x^2+\sqrt{xy-3y},$$

are functions of x and y; and

$$x+y+z, \quad y^2+\tan\frac{x}{y}, \quad z\sin^2(x^2y), \quad \sqrt[3]{x^2+y^2}+\log z,$$

are functions of x, y and z. Each variable is independent of the others. Particular values or expressions may be assigned to one or more of the variables, and the result discussed as a function of the remaining variables. A function of two or more variables possesses all of its properties as a function of each variable. By substituting in the function $2x^2+y$, any assumed value for y, as 5, the result $2x^2+5$ is a function of a single variable.

7. A quantity is a function of the *sum of two variables* when every operation indicated upon either variable includes the *sum* of the two. Thus,

$$3c\sqrt{x\pm y},\qquad \sin(x\pm y),\qquad \log(x\pm y),\qquad a^{x\pm y},$$

and all algebraic expressions which may be written in the form

$$A(x\pm y)^n+B(x\pm y)^{n-1}+\ldots\ldots +H,$$

in which A, B, etc., are constants, are functions of the sum of the two variables x and $\pm y$.

$$8ax(x+y)^n,\qquad \sqrt{x-y}-2y,\qquad \sqrt{x}+y,\qquad x^{x+y},\qquad x\sin(x-y),$$

are not functions of the sum of x and y.

$$\sin(x^2\pm y^2),\qquad A(x^2\pm y^2)^n,\qquad 3\log(x^2\pm y^2),\qquad \sqrt[3]{2(x^2\pm y^2)}+7a,$$

are functions of the *sum* of the two variables x^2 and $\pm y^2$, but not of the sum of x and $\pm y$.

$$2(b\sqrt{x}+ay^2),\qquad \cos^2(b\sqrt{x}+ay^2),\qquad 2\sqrt{\log(b\sqrt{x}+ay^2-3c)},$$

are functions of the sum of the two variables $b\sqrt{x}$ and ay^2.

In any function of the sum of two variables, a single variable may be substituted for the sum, and the original function expressed as a function of the new variable. Thus, z may be substituted for $(x+y)$ in the function $a(x+y)^n$, giving the function in the form az^n. In a similar manner we may write

$$\tan(x-y)=\tan z,\qquad a^{x+y}=a^z,\qquad 2a\sqrt{\log(x-y)}=2a\sqrt{\log z}\,;$$

but it must be remembered that z in the new form is a function of the two variables x and y.

8. A *state* of a function of two or more variables, corresponding to a set of values or expressions for the variables, is the *result* obtained by substituting those values or expressions for the corresponding variables. Thus,

$$-20,\qquad -6,\qquad 0,\qquad 5,\qquad 25,$$

are states of the function $4x+3y+2$ corresponding, respectively, to the values or expressions for x and y,

$$(-4,-2), \quad (-2,0), \quad (-8,+10), \quad (0,1), \quad (2,5);$$

and

$$0, \quad \sqrt{\frac{1}{3}}, \quad 1, \quad \sqrt{3}, \quad \infty,$$

are states of the function $\tan(x+y)$ corresponding, respectively, to the values or expressions for x and y,

$$(0,0), \quad \left(\frac{\pi}{18}, \frac{\pi}{9}\right), \quad \left(\frac{\pi}{12}, \frac{\pi}{6}\right), \quad \left(\frac{2\pi}{9}, \frac{\pi}{9}\right), \quad \left(0, \frac{\pi}{2}\right).$$

Any function, in which all of the variables are independent, is a variable, and has an infinite number of states.

9. A function of several variables may be equal to some constant value or expression; in which case one of the variables is dependent upon the others. Thus, the first member of the equation $2x+3y=7$ is a function of the two variables, x and y; but x and y are mutually dependent.

Any equation containing n variables expresses a dependence of each variable upon the others; and there are only $n-1$ *independent* variables in such an equation. In other words, the number of *independent* variables in any equation is one less than the total number of variables.

In any group of simultaneous equations, the number of *independent* variables is equal to the total number of variables less the number of independent equations.

10. Functions are divided into two general classes, *abstract* and *concrete.*

Abstract functions are subdivided into *algebraic* and *transcendental.*

11. An **Algebraic** function is one that can be expressed *definitely* by the ordinary operations of Algebra; that is, by addition, subtraction, multiplication, division, formation of powers with constant exponents, and extraction of roots with constant indices.

12. Certain algebraic functions have particular names based upon peculiarities of form.

A *rational* function of a variable is one in which the variable is not affected by a fractional exponent.

An *integral* function of a variable is one in which the variable does not enter the denominator of a fraction, or in other words, is not affected by a negative exponent.

$$x^m+Ax^{m-1}+Bx^{m-2}+\quad.\ .\ .\ .\ .\ .\ .\quad Gx+H,$$

in which m is a positive integer, and A, B, etc., do not contain x, is a *rational* and *integral* function of x. The coefficients A, B, etc., may be irrational or fractional.

A rational integral function of a variable is also called an *entire* function of that variable.

A *linear* function of two or more variables is one in which each term is of the first degree with respect to the variables.

Thus, $2x+3y+7z$ is a linear function of x, y and z.

A function is homogeneous with respect to its variables when each term is of the same degree with respect to them.

A linear function is a homogeneous function of the first degree.

13. Other divisions of functions, based upon form or properties, are of frequent use.

Explicit and Implicit Functions. When a function is expressed directly in terms of its variable or variables, it is an explicit function; otherwise it is an implicit function.

Thus, in the equations

$$y=2x^2+3z,\qquad y=\tan^2 x,\qquad y=3^x,\qquad y=\log 2ax^2,\qquad y=f(x,z),$$

y is an explicit function of the variables in the second members, and in the equations

$$a^2y^2+b^2x^2=a^2b^2,\quad y^{\frac{2}{3}}=\log x^2,\quad y^2=r^2-x^2,\quad \sqrt{y}=\cot x,\quad y^n=f(x),\quad f(y,x)=0,$$

y is an implicit function of x.

The relation between an implicit function and its variables is sometimes expressed by two equations. Thus, $y=3u$, $u^2=\sqrt{x}$, in which y is an implicit function of x.

$$y=f(u),\ u=\varphi(x);\quad \text{and}\quad y=f(u),\ x=\varphi(u),$$

are forms expressing y as an implicit function of x.

14. Increasing and Decreasing Functions. A function that increases when a variable increases, and decreases when that variable decreases, is an increasing function of that variable. Thus, $2x$, $7x^3$, 2^x, $\frac{ax^3}{b}$, are increasing functions of x.

A function that decreases when a variable increases, and increases when that variable decreases, is a decreasing function of that variable. Thus, $\frac{1}{x}$, $(c-x)^3$, $\frac{b}{ax^3}$, are decreasing functions of x.

Functions are sometimes increasing for certain values of the variable, and decreasing for others. Thus, $(c-x)^2$ is an increasing function for all values of x greater than c; but decreasing for all values of x less than c. $2ax$ is an increasing function when x is positive, and decreasing when x is negative. The positive value of $y=\pm\sqrt{r^2-x^2}$, is an increasing function for values of x from $-r$ to 0, but decreasing for values of x from 0 to $+r$. The negative value of y is a decreasing function for negative values of x, and increasing for positive values of x.

15. Continuous and Discontinuous Functions. A function is continuous between states corresponding to any two values of a variable, when it has a real state for every intermediate value of the variable; and as the difference between any two intermediate values of the variable approaches zero, the difference between the corresponding states approaches the same limit. Otherwise a function is discontinuous between the states considered.

A continuous function in passing from any assumed state to another must pass through all states intermediate to those assumed; but it may have intermediate states greater or less than the states assumed. Thus, the function $\sqrt{r^2-x^2}$ is continuous between the states 0 and $\frac{r}{2}\sqrt{3}$, which correspond to $x=-r$ and $x=\frac{r}{2}$; but it is greater when $x=0$ than either of the states considered.

An imaginary or infinite state, or the omission of any state between the extreme states considered, interrupts the continuity.

A function always continuous changes its sign only by passing through zero; but a discontinuous function may change its sign without passing through zero.

Entire functions of a variable are always continuous.

$\pm\sqrt{2px}$ is continuous between states corresponding to $x=-0$ and $x=\infty$.

$\pm\frac{b}{a}\sqrt{a^2-x^2}$ is continuous between states corresponding to $x=-a$ and $x=+a$.

$\pm\frac{b}{a}\sqrt{x^2-a^2}$ is continuous between states corresponding to $\begin{cases} x=-\infty \text{ and } x=-a. \\ x=a \text{ and } x=\infty. \end{cases}$

but is discontinuous between states corresponding to $x=-a$ and $x=a$.

16. A **Transcendental** function is one that cannot be expressed *definitely* by the ordinary operations of Algebra.

In general, a transcendental function may be expressed algebraically by an infinite series.

Transcendental functions are of four kinds, *exponential*, *logarithmic*, *trigonometric*, and *inverse trigonometric*.

An **Exponential** function is one with a variable exponent; as,

$$a^x, \qquad \left(\frac{a}{x}+1\right)^x, \qquad e^x-2cx.$$

A **Logarithmic** function is one that contains a logarithm of a variable; as,

$$\log x, \qquad \log(a+y), \qquad 2ax^2-\frac{x}{\log x}.$$

A **Trigonometric** function is one that involves the sin, or cos, or tan, or cosec, etc., of a variable angle; as,

$$\cot x, \qquad \sec 2x^2, \qquad \frac{x-\sin x}{x^3}, \qquad \frac{\tan x - x}{x-\sin x}, \qquad \text{versin}^2 x.$$

An **Inverse Trigonometric** function is one that contains an angle regarded as a function of a variable sin, or cos, or tan, etc.

$\sin^{-1}y$, $\tan^{-1}y$, $\text{cosec}^{-1}y$, read "the angle whose sin is y"; "whose tan is y"; "whose cosec is y"; are symbols used to denote such functions. Having given $y=\text{versin}\,x$, then $x=\text{versin}^{-1}y$; and if $u=\cos y$, then $y=\cos^{-1}u$, etc.

17. It should be observed that although the number of different abstract functions of a single variable is infinite, they involve but ten elementary or simple forms, five of which are the inverse of the others.

Representing the direct functions by y, and the variable by x, the forms are as follows:

Algebraic.

Direct.		*Inverse.*	
$y=x+a$.	Sum.	$x=y-a$.	Difference.
$y=ax$.	Product.	$x=\frac{y}{x}$.	Quotient.
$y=x^n$.	Power.	$x=\sqrt[n]{y}$.	Root.

Transcendental.

$y=a^x$.	Exponential.	$x=\log_a y$.	Logarithmic.
$y=\sin x$.	Trigonometric.	$x=\sin^{-1} y$.	Inverse Trigonometric.

18. Functional Notation. A function of any quantity, as x, is represented thus, $f(x)$, read "function of x". Other forms are also used, as, $f'(x)$, $F(x)$, $F_1(x)$, $\varphi(x)$, $\varphi^1(x)$, $\psi(x)$, $\psi_1(x)$. The quantity is written within the brackets, and a letter, as f, or F, or φ, etc., is placed before the brackets to represent the operations involved in any particular function.

Having assumed the exterior letter, its significance remains unchanged throughout the same subject. Thus, if $\frac{ax}{1+x}$ is represented by $f(x)$, f indicates that the quantity within the brackets is multiplied by a, and that the product is divided by 1 plus the quantity. Hence,

$$f(y)=\frac{ay}{1+y},\quad f(z)=\frac{az}{1+z},\quad f(m)=\frac{am}{1+m},\quad f(2)=\frac{2a}{1+2},\quad f(\sin\varphi)=\frac{a\sin\varphi}{1+\sin\varphi}.$$

Like functions of different quantities, when considered in the same subject, require the same exterior letter. In order to represent different functions of the same quantity, the exterior letter is changed, but not the letter within the brackets. Thus, if $F(x)$ is selected to represent $2\sqrt{bx}$; then some other form, as $F_1(x)$, or $\varphi(x)$, etc., should be taken to denote $4cx^3+2x$.

Different functions of different quantities are represented by forms which have different letters within and without the brackets. Thus, $\sqrt{x^2-a^2}$, and $\frac{4y^2}{1-y^2}$, may be denoted by $f(x)$, and $\psi(y)$, respectively.

A function of x^2 is written $f(x^2)$, or $F(x^2)$, etc., and the square of a function of x is designated by $\overline{f(x)}^2$, or $\overline{\varphi(x)}^2$, etc.

$3c\sqrt{my^2}$ may be expressed as a function of my^2 by some form, as $f(my^2)$, or $f'(my^2)$, etc.

Having represented az^2 by $f(z)$, and $3c\sqrt{az^2}$ by $F(az^2)$, we may write $3c\sqrt{az^2}=F(az^2)=F[f(z)]$.

Similarly, having $a^x=\varphi(x)$, and $b\sqrt{a^x}=\psi(a^x)$, we write

$$\frac{8b\sqrt{a^x}-3eb\sqrt{a^x}}{2db\sqrt{a^x}}=f'\Big(\psi\big[\varphi(x)\big]\Big),\ \text{in which}\ \psi\big[\varphi(x)\big]=b\sqrt{a^x}.$$

An expression containing several different functions of a variable, as $2ax^2-\log x+3\sin x$, may be considered as a function of the several functions of the variable, and represented thus,

$$F[f(x),\ \varphi(x),\ \psi(x)].$$

Different functions of the same variable, as x, are frequently denoted by the symbols X, X', X'', etc.; in which case a function of the several functions may be indicated by $f(X, X', X'', \text{etc.})$.

$\varphi\left[F(y), \psi(x), f(z)\right]$ represents a function of three different functions of different variables.

Representing different functions of x by X, X', X'', and various functions of y by Y, Y', Y'', a function of the several functions of x and y would be indicated by $F(X, X', X'', Y, Y', Y'')$.

Functions of two variables are denoted thus, $f(x,y)$, $f'(x,y)$, $F(y,z)$, $\varphi(x,y)$, $\psi(x,z)$, $\psi_1(x,z)$, etc.; and functions of three variables by $F(x,y,z)$, $\psi(r,s,t,)$, etc.

Functions of any number of variables are indicated similarly by writing all the variables within the brackets or parenthesis.

In all cases the like exterior symbols have the same significance in any one subject.

Thus, if $f(x,y)=ax+by$; then $f(s,t)=as+bt$; $f(2,3)=2a+3b$; $f(0,m)=bm$.

Having $\varphi(x,y,z)=2x-cz+y^2$; then $\varphi(r,s,t)=2r-cs+t^2$.

Functions are frequently represented by single letters; thus $\pm\sqrt{R^2-x^2}$ may be represented by y, giving $y=\pm\sqrt{R^2-x^2}$; and $f(x,y)$ by z, giving $z=f(x,y)$.

Illustrations.

1. Having $f(x)=x^m+Px^{m-1}+Qx^{m-2}+\dots+U$, in which P, Q, etc., do not contain x; then,

$$f(5)=5^m+P5^{m-1}+Q5^{m-1}+\dots+U.$$
$$f(3bc)=(3bc)^m+P(3bc)^{m-1}+\dots+U.$$
$$f(a-x)=(a-x)^m+P(a-x)^{m-1}+\dots+U.$$
$$f(o)=o^m+Po^{m-1}+\dots+U.$$
$$f(c^2)=(c^2)^m+P(c^2)^{m-1}+\dots+U.$$

	Having	then,
2.	$\varphi(a)=4a^2+ca$;	$\varphi(x+y)=4(x+y)^2+c(x+y)$,
3.	$\psi(ax^3)=4(ax^3)^2+c(ax^3)$;	$\psi(\sin\theta)=4\sin^2\theta+c\sin\theta$.
4.	$F(x)=a^x$;	$F(x+y)=a^{x+y}=a^x\times a^y=F(x)\times F(y)$.
5.	$F(xy)=a^{xy}$;	$\overline{F(x)}^{\,y}=\overline{F(y)}^{\,x}=(a^x)^y=(a^y)^x=F(xy)$.

6. If $\varphi(z)=\sqrt[n]{z}$; $\psi(x)=5ax$; and $F(w)=\dfrac{c\sqrt{w}-(w)^{\frac{1}{2}}}{3w-2p}$;

then $$F\Big(\psi\left[\varphi(z)\right]\Big)=\frac{c\sqrt{5a\sqrt[n]{z}-\left(5a\sqrt[n]{z}\right)^{\frac{1}{2}}}}{15a\sqrt[n]{z}-2p}=f(z).$$

Having	then,	
7. $f(x,y)=ax-by$;	$f(y,x)=ay-bx$;	and $f(z,z)=az-bz$.
8. $\psi(y,z)=a^y+\sqrt[n]{z}$;	$\psi(a,b)=a^a+\sqrt[n]{b}$;	and $\psi(x,y)=a^x+\sqrt[n]{y}$.
9. $\varphi(x,y,z)=3x-\log y+\tan z$;	$\varphi(r,s,t)=3r-\log s+\tan t$.	
10. $F(x,y,z)=x^3+2y+\sqrt{z}$;	$F(2,-8,64)=0$;	and $F(0,0,16)=4$.

11. $\dfrac{3a^x-\tan\theta+2y^n}{7}=\psi\left[f(x), F(\theta), \varphi(y)\right]$; in which, $f(x)=\dfrac{3a^x}{7}$;

$F(\theta)=\dfrac{\tan\theta}{7}$; $\varphi(y)=\dfrac{2y^n}{7}$.

12. $f(x,y,z)=m^{ax+by+cz}=m^{ax}\times m^{by}\times m^{cz}=f(x,0,0)\times f(0,y,0)\times f(0,0,z)$.

13. If $\varphi(x,y)=2x+\sin y$; and $\psi(z)=3\sqrt{z}$; then $\psi\left[\phi(x,y)\right]=3\sqrt{2x+\sin y}$.

14. If $f(x,y,z)=7ax^2yz$; and $F(y)=\sqrt[3]{y^2}$; and $\varphi(x)=a^x$; and $\psi(z)=2z$;

then $$\psi\left[\varphi\left(F\left[f(x,y,z)\right]\right)\right]=2a^{\sqrt[3]{(7ax^2yz)^2}}.$$

19. Lines. Any portion of any line may be considered as generated by the continuous motion of a point.

Let s represent the length of a varying portion of any line in the coördinate plane XY, of which the equation in x and y is given. s depends upon the coördinates of its variable extremities, and varies with each; but the equation of the line establishes a dependence between these coördinates. Hence, s is a function of one *independent* variable only.

If the line is in space, its two equations establish a dependence between the three coördinates of its extremities, so that one only is independent.

The same result will follow if a system of polar coördinates is used.

20. Geometric Representative of a Function of a Single Variable. By laying off upon the axis of abscissas assumed values of any variable, and upon the corresponding ordinates, distances representing the corresponding states of any given function of that variable, a line may be determined, the coördinates of whose points will have the same relations as those existing between the corresponding states of the function and values of the variable.

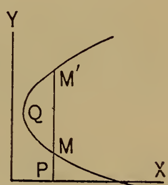
Hence, *every function of a single variable may be geometrically represented by the variable ordinate of a line, of which the variable abscissa represents the variable.*

It follows, that the relation between a function and its variable may be expressed analytically by the equation formed by placing the function equal to a symbol representing the varying ordinate. Thus, placing the function $7x^2+3$ equal to y, we have $y=7x^2+3$, which expresses the relations between the variable coördinates y and x, and therefore between the function $7x^2+3$, and its variable x.

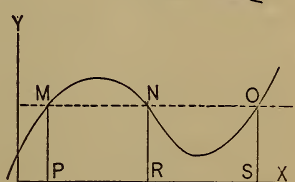
The equation thus obtained is that of a line whose **ordinate,** not the line, represents geometrically the given function.

A function of a single variable, which is of the first degree with respect to the variable, will be represented geometrically by the ordinate of a right line.

The ordinates PM, and PM' of the curve MQM', represent geometrically two different states of the function corresponding to the same value of the variable, § 5.

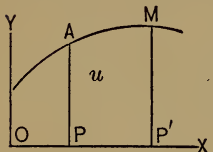


The ordinates PM, RN, and SO of the curve MNO, represent geometrically equal states of the function corresponding to different values of the variable, § 5.



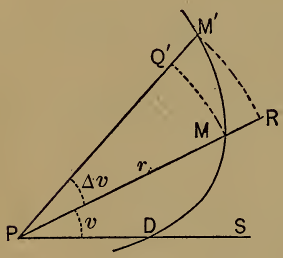
It is important to notice that the function represented by a line is not, in general, the function represented by its ordinate. The problem of determining a line which represents a given function of a single variable; or a function which is represented by a given line, is not, in general, a simple one. Therefore, the method of representing geometrically a function of a single variable by the ordinate of a line is generally adopted.

21. Surfaces. Any portion of any surface may be considered as generated by the continuous motion of a line.

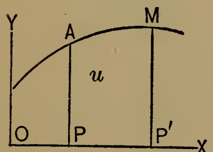


Let u represent the area of a varying portion of the surface generated by the continuous motion of the ordinate of any given line in the plane XY.

u depends upon the coördinates of the variable extremities of that portion of the given line which limits it, and varies with each; but the equation of the given line establishes a dependence between these coördinates. Hence, u is a function of but one *independent* variable.



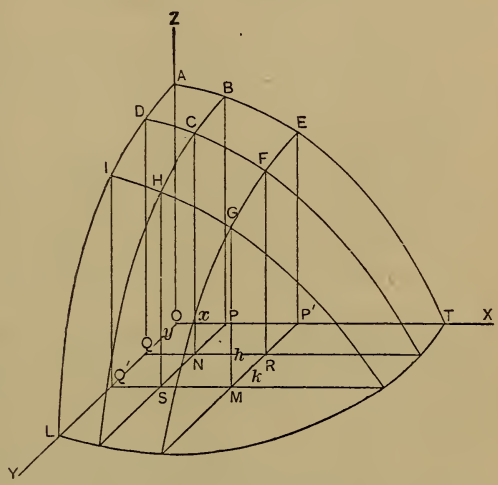
22. Let $r = f(v)$ be the polar equation of any plane curve, as DM, referred to the pole P, and the right line PS. Let u represent the area of a varying portion of the surface, generated by the radius vector revolving about the pole. u will change with v and r; but v and r are mutually dependent. Hence, u is a function of but one *independent* variable.



23. Let any line in the plane XY, as AM, revolve about the axis of X. It will generate a surface of revolution.

The same surface may be generated by the circumference of a circle, whose centre moves along the axis X, with its plane perpendicular to it; and whose radius changes with the abscissa of the circle, so as to always equal the corresponding ordinate of the curve AM. The radius of the generating circumference is, therefore, a function of the abscissa of its centre. Hence, the generating circumference, and any varying portion of the surface generated, is a function of but one *independent* variable.

24. The area of any surface with two independent variable dimensions is a function of two independent variables. For example, the area of any rectangle with variable sides, parallel respectively to the coördinate axes X and Y, is a function of the two independent variables x and y.



25. Having any surface, as ATL, let $ABCD = u$ be a portion included between the coördinate planes XZ, YZ, and the planes DQR and BPS, parallel to them respectively. Let $OP = x$, and $OQ = y$ be independent variables. u will depend upon x, y and z; but the equation of the surface makes z dependent upon x and y. Hence, u is a function of but two *independent* variables. Similarly, it may be shown that any varying portion of the surface included between any four planes, parallel two and two, to the coördinate planes XZ and YZ, is a function of but two *independent* variables.

26. Geometric Representative of a Function of Two Variables. By laying off upon the axes of x and y, respectively, assumed values of any two variables; and upon the corresponding ordinates, distances representing the corresponding states of any given function of the two variables, a surface may be determined having the same relations between the coördinates of its points, as those existing between the corresponding states of the function and values of the variables.

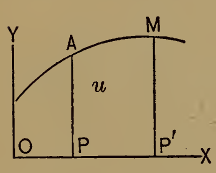
Hence, *every function of two variables may be geometrically represented by the variable ordinate of a surface, of which the variable abscissas represent the variables.*

It follows, that the relations between such a function and its variables may be expressed analytically by the equation formed by placing the function equal to a letter representing the varying ordinate. Thus, placing the function $2x^2+5y+7$ equal to z, we have $z=2x^2+5y+7$; which expresses the relations between the variable coördinates z, x and y, and therefore between the function $2x^2+5y+7$, and its variables x and y. The equation thus obtained is that of a surface whose **ordinate** represents the given function geometrically.

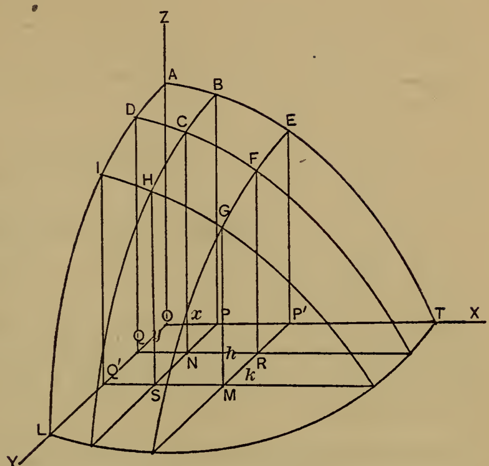
It is important to notice that the function represented by a surface is not, in general, the function represented by the ordinate of the surface.

The problem of determining a surface which represents a given function of two variables; or a function which is represented by a given surface, is not, in general, a simple one. Therefore, the method of representing geometrically a function of two variables by the variable ordinate of a surface is generally adopted.

27. Volumes. Any portion of any volume may be considered as generated by the continuous motion of a surface. The form of the surface, and the law of its motion determine the nature and class of the volume.



Let any plane surface included between any line in the plane $X\ Y$, as $A\ M$, and the axis of X be revolved about X. It will generate a volume of revolution. The same volume may be generated by the circle, whose centre moves along the axis X, with its plane perpendicular to it; and whose radius changes with the abscissa of the circle, so as to always equal the corresponding ordinate of the curve AM. The radius of the generating circle is, therefore, a function of the abscissa of its centre. Hence, the generating circle, and any varying portion of the volume generated, is a function of but one *independent* variable.



28. Having any volume, as ATL, bounded by a surface whose equation is given, and the coördinate planes, let $ABCD - ON = V$, be a portion included between the coördinate planes XZ, YZ, and the planes DQR and BPS, parallel to them respectively.

Let $OP = x$, and $OQ = y$, be independent variables. V will depend upon x, y and z; but the equation of the surface makes z dependent upon x and y. Hence, V is a function of but two *independent* variables.

In a similar manner, it may be shown that any varying portion of the volume included between any four planes, parallel two and two, to the coördinate planes XZ and YZ, is a function of but two independent variables.

29. Any volume with three independent variable dimensions is a function of three independent variables. For example, the volume of any parallelopipedon with variable edges parallel, respectively, to the coördinate axes, X, Y and Z, is a function of x, y and z; all of which are independent.

CHAPTER II.

PRINCIPLES OF LIMITS.

30. The Limit of a variable* is a fixed quantity or expression which the variable, in accordance with a law of change, continually approaches but never equals; and from which it may be made to differ by a quantity less numerically than any assumed quantity however small.

As an example, take the variable expression $\frac{x}{1+x}$, and increase x continually. Under this law, $\frac{x}{1+x}$ will continually approach, but never equal, unity. x may be taken so large that the difference between the corresponding value of $\frac{x}{1+x}$ and unity, will be less numerically than any assumed number however small. Unity is therefore the limit of the variable $\frac{x}{1+x}$, under the law that x increases and becomes greater than any assumed number. This may be indicated as follows,

$$\underset{x \to \infty}{\text{limit}} \left[\frac{x}{1+x}\right] = 1, \quad \text{or} \quad \frac{\infty}{1+\infty} = 1;$$

or, having represented $\frac{x}{1+x}$ by $f(x)$, we may write $f(\infty) = 1$.

$\frac{3}{11}$ is the limit of the repeating decimal fraction $0.272727.....$, under the law that the number of places of figures is indefinitely increased.

The circumference of a circle is the limit of the perimeter of an inscribed regular polygon as the number of its sides is continually increased. The radius is the limit of the apothem, and the circle, of the polygon, under the same law.

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* In this chapter the term variable is used in its general sense § 1, and includes all functions of variables.

In all cases, when referring to the limit of a variable, it is necessary to give the law; for the limit depends not only upon the variable, but also upon the law by which it changes. Under a law, a variable has but one limit; but it may have different limits under different laws.

The manner in which a variable approaches a limit depends upon the law. It may be less than the limit, and continually increase; or it may be greater than the limit, and continually decrease. A variable sometimes approaches a limit by values alternately greater and less than the limit.

Examples of the latter class may be found in the n successive approximating fractions of a continued fraction. The continued fraction being the limit of the successive approximating fractions under the law that n increases.

0 is the limit of any number, which under a law of change, becomes less numerically than any assumed number however small.

∞ is the limit of any number, which under a law of change, becomes greater numerically than any assumed number however great.

0 and ∞ are neither numbers nor measures of quantities.

Limits, as defined, include all results obtained by the substitution of 0 or ∞ for any variable quantity or quantities which enter any expressions. Thus, A is the limit of $A+h$ as h approaches 0.

Since infinity is indefinite, two infinities cannot, in general, be compared with each other.

Expressions, such as

$$\frac{a}{0}=\infty, \qquad \frac{\infty}{1+\infty}=1, \qquad f(\infty)=1,$$

are symbolic forms indicating the limits of certain variables, and the law of change.

The statement that one number or value is infinitely great as compared with another, is inaccurate. A number, however small, cannot be neglected or omitted in comparison with any other, however great, without error. In applied mathematics numbers or values are sometimes neglected in comparison with others when approximate results are sufficiently accurate for the object in view.

Any variable, which under a law approaches zero as a limit, is called an *infinitesimal.*

As any variable approaches its limit, under a law, the difference between it and the limit approaches zero as a limit. Hence, the difference between any variable and its limit is an infinitesimal under the same law.

Let u be any variable, and C a constant which is its limit under a law. Let ε represent the infinitesimal $C-u$; then,

$\varepsilon = C - u \ldots (1)$, or $u = C - \varepsilon \ldots (2)$, or $C = u + \varepsilon \ldots (3)$,

in which the sign of ε depends upon C and u.

That is,

1°. A constant is the limit of a variable when the difference between the constant and the variable is an infinitesimal under the law.

2°. A variable is always equal to its limit under a law, minus the infinitesimal which is the variable remainder obtained by subtracting the variable from its limit.

3°. A constant is the limit of a variable when it is the sum of the variable and an infinitesimal under the law.

An infinitesimal is not necessarily a small quantity in any sense. Its essence lies in its power of decreasing numerically, in other words, in having zero as a limit; and not in any small value that it may have. It is frequently defined as "*an infinitely small quantity*"; that is not, however, its significance as here used.

In representing infinitesimals by geometric figures they should be drawn conveniently large; and it is useless to strain the imagination in vain efforts to conceive of the appearance of the figure when the infinitesimals decrease beyond our perceptive faculties. Usually one or two auxiliary figures representing the magnitudes at one or two of their states under the law, give all the assistance that can be derived from figures.

31. Theorem I. *A variable with a constant sign cannot have a limit with a contrary sign.*

For suppose $f(x)$ is always positive, and that *limit* $f(x) = -C$. From the definition of a limit, § 30, $f(x)$ may be made to differ from $-C$ by a value numerically less then C. It would therefore become negative, which is contrary to the hypothesis. In a similar manner, it may be shown that a variable always negative cannot have a positive limit.

Theorem II. *If the corresponding values of any two variables approaching their limits under a law, are always equal, the variables have the same limit.*

Let u and v represent any two variables giving always, under a law, $u = v$.

Suppose C to be the limit of u; then $u = C - \varepsilon$, in which ε is an infinitesimal under the law.

Substituting in above we have

$$C - \varepsilon = v, \quad \text{or} \quad C = v + \varepsilon.$$

Hence, C is the limit of v under the same law.*

Theorem III. *If the difference between the corresponding values of any two variables, approaching their limits under a law, is an infinitesimal, the variables have the same limit.*

Let u and v represent any two variables giving

$$u - v = \delta, \quad \text{or} \quad u = v + \delta,$$

in which δ is an infinitesimal.

Let C be the limit of u, then $u = C - \varepsilon$, in which ε is an infinitesimal.

Substituting in above we have

$$C - \varepsilon = v + \delta, \quad \text{or} \quad C - v = \delta + \varepsilon,$$

the second member of which is an infinitesimal. Hence, C is the limit of v.

Theorem IV. *The limit of the sum or difference of any number of variables is the sum or difference of their limits.*

Let u, v, w, etc., represent any variables, and A, B, C, etc., their respective limits; then

$$u = A - \varepsilon, \qquad v = B - \delta, \qquad w = C - \omega, \text{ etc.},$$

in which ε, δ, ω, etc., are infinitesimals.

Adding, or subtracting, the corresponding members we have

$$\pm u \pm v \pm w \pm \text{\&c.} = \pm A \pm B \pm C \pm \text{\&c.} \mp \varepsilon \mp \delta \mp \omega \mp \text{\&c.}$$

Hence, Theorem II,

$$\begin{aligned} \text{limit}\,[\pm u \pm v \pm w \pm \text{\&c.}] &= \pm A \pm B \pm C \pm \text{\&c.} \\ &= \pm \text{limit}\, u \pm \text{limit}\, v \pm \text{limit}\, w \pm \text{\&c.} \end{aligned}$$

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* Hereafter, in order to avoid the frequent repetition of the expression "under the law", it will be assumed, unless otherwise stated, that the changes in all the variables considered together, or in the same theorem, are due to one and the same law; and that all variables and their functions are continuous between all states considered.

Theorem V. *The limit of the product of any two variables with finite limits, is the product of their limits.*

Let u and v represent any two variables having the finite limits A and B respectively; then

$$u = A - \varepsilon, \quad \text{and} \quad v = B - \delta,$$

in which ε and δ are infinitesimals.

Multiplying, member by member, we have

$$uv = AB - B\varepsilon - A\delta + \varepsilon\delta.$$

Hence, Theorem II,

$$\text{limit}\,[u\,v] = A\,B = \text{limit}\ u.\ \text{limit}\ v.$$

Cor. *The limit of any power or root of any variable with a finite limit, is the corresponding power or root of its limit.*

Thus,

$$\text{limit}\ u^m = (\text{limit}\ u)^m, \quad \text{and} \quad \text{limit}\ u^{\frac{1}{m}} = (\text{limit}\ u)^{\frac{1}{m}}.$$

Theorem VI. *The limit of the quotient of any two variables with finite limits, is, in general, the quotient of their limits.*

With the same notation as above we obtain by division,

$$\frac{u}{v} = \frac{A-\varepsilon}{B-\delta} = \frac{A}{B} + \frac{A\delta - B\varepsilon}{B(B-\delta)}.$$

Hence, if limit $v = B$ is not zero, we have

$$\text{limit}\left[\frac{u}{v}\right] = \frac{A}{B} = \frac{\text{limit}\ u}{\text{limit}\ v}.$$

If both u and v are infinitesimals, the theorem fails; as it should since limit $\left[\frac{u}{v}\right]$ can have but one value, § 30; whereas $\frac{\text{limit}\ u}{\text{limit}\ v} = \frac{0}{0}$, may have an infinite number of values.

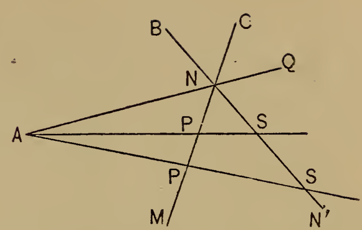
Hence, we *cannot* write limit $\left[\frac{u}{v}\right] = \frac{\text{limit}\ u}{\text{limit}\ v} = \frac{0}{0}$.

This failing case of the theorem is particularly important, as it explains a subsequent result upon which the main application of the principles of limits to the Calculus is based. Some examples are given to illustrate it.

1°. $$\underset{e \ggg 0}{\text{limit}}\left[\frac{\sqrt{1+e}-1}{e}\right] = \underset{e \ggg 0}{\text{limit}}\left[\frac{(\sqrt{1+e}-1)(\sqrt{1+e}+1)}{e(\sqrt{1+e}+1)}\right]$$

$$= \underset{e \ggg 0}{\text{limit}}\left[\frac{1+e-1}{e(\sqrt{1+e}+1)}\right] = \underset{e \ggg 0}{\text{limit}}\left[\frac{1}{\sqrt{1+e}+1}\right] = \frac{1}{2};$$

whereas $$\frac{\underset{e \ggg 0}{\text{limit}}\left[\sqrt{1+e}-1\right]}{\underset{e \ggg 0}{\text{limit}}\left[e\right]} = \frac{0}{0}.$$

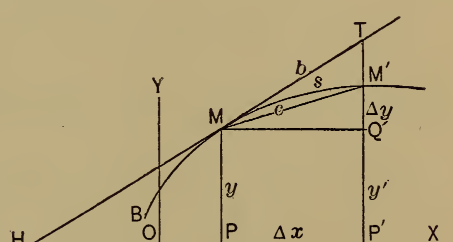


2°. Through the point A, without the angle MNN', draw right lines APS intersecting the sides MN and $N'N$ nearer and nearer to N. The segments PN and SN are infinitesimals under the law, and we have $\dfrac{\text{limit } PN}{\text{limit } SN} = \dfrac{0}{0}$.

The ratio $\dfrac{PN}{SN}$ is always equal to the corresponding value of $\dfrac{\sin NSP}{\sin NPS}$.

Hence, limit $\left[\dfrac{PN}{SN}\right] =$ limit $\left[\dfrac{\sin NSP}{\sin NPS}\right] = \dfrac{\sin BNA}{\sin CNQ}$, which is determinate.

The following example not only illustrates the case under consideration, but it also establishes a principle of great importance.



3°. Represent any function of any single variable, as x, by y, giving $y = f(x)$.

Let BMM' be the curve whose ordinate represents the given function, § 20. Take any state of the function, as PM corresponding to $x = OP$, and increase x by PP' represented by Δx. Draw the ordinate $P'M'$ and the secant MM'. Through M draw MQ' parallel to X. $Q'M'$, denoted by Δy, will represent the increment of the function corresponding to Δx.

$\dfrac{Q'M'}{PP'} = \dfrac{\Delta y}{\Delta x} = \tan Q'MM'$ will be the ratio of the increment of the function to the corresponding increment of the variable.

At M draw MT tangent to the curve. Then, under the law that Δx approaches zero, the secant MM' will approach coincidence with the tangent MT, and the angle $Q'MM'$ will approach the angle $Q'MT$, or its equal XHT, as a limit.

Hence,

$$\underset{\Delta x \rightarrow 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \underset{\Delta x \rightarrow 0}{\text{limit}} [\tan Q'MM'] = \tan XHT.$$

That is, *the limit of the ratio of any increment of any function of a single variable to the corresponding increment of the variable, under the law that the increment of the variable approaches zero, is equal to the tangent of the angle made with the axis of abscissas by a tangent, to the line whose ordinate represents the function, at the point corresponding to the state considered.*

When M' coincides with M the secant may have any one of an infinite number of positions other than that of the tangent line $M\ T$, for the only condition then imposed is that it shall pass through M. Therefore, while $\underset{\triangle x \Rightarrow 0}{\text{limit}}\left[\frac{\triangle y}{\triangle x}\right]$ is definite, and equal to the tangent of the angle that the tangent line at M makes with X, $\frac{\text{limit } \triangle y}{\text{limit } \triangle x} = \frac{0}{0}$ indicates that the tangent of the angle which the secant makes with X becomes indeterminate when M' coincides with M.

Limit $\left[\frac{\triangle y}{\triangle x}\right]$ is, therefore, one of the many values that $\frac{\text{limit } \triangle y}{\text{limit } \triangle x}$ may have under the law.

It should be observed that if limit $\left[\frac{\text{u}}{\text{v}}\right] = 1$, then limit u $=$ limit v; but having limit u $=$ limit v, it does not follow that limit $\left[\frac{\text{u}}{\text{v}}\right] = 1$, unless the limit of each is finite and not zero.

Theorem VII. *The limit of the logarithm of any variable with a finite limit, is the logarithm of the limit of the variable.*

Let $(1+y)$ represent any variable with a finite limit.

From Algebra we have

$$\log(1+y) = M\left[y - \frac{y^2}{2} + \frac{y^3}{3} - \&c.\right].$$

Hence, Theorem II,

$$\text{limit} \log(1+y) = M\left[\text{limit } y - \frac{(\text{limit } y)^2}{2} + \frac{(\text{limit } y)^3}{3} - \&c.\right] = \log(1 + \text{limit } y).$$

Theorem VIII. Limit $a^x = a^{\text{limit } x}$.

From Algebra we have

$$a^x = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \&c.,$$

in which c_1, c_2, &c., are constants, respectively, equal to $\log_e a$, $\frac{(\log_e a)^2}{2}$, &c.

Hence, Theorem II,

$$\text{limit } a^x = 1 + c_1 \text{ limit } x + c_2 (\text{limit } x)^2 + \&c. = a^{\text{limit } x}.$$

Theorem IX. Limit $\sin \psi = \sin \text{ limit } \psi$.

From Trigonometry we have

$$\sin \psi = \psi - \frac{\psi^3}{1.2.3} + \frac{\psi^5}{1,2.3.4.5} - \&c.$$

Hence, Theorem II,

$$\text{limit } \sin \psi = \text{limit } \psi - \frac{(\text{limit } \psi)^3}{1.2.3} + \frac{(\text{limit } \psi)^5}{1.2.3.4.5} - \&c. \ \sin = \sin \text{ limit } \psi.$$

From the preceeding theorems we learn, that, in general, *the limit of any continuous function of one or more variables is the same function of their respective limits under the law.*

That is,

$$\text{limit } f\,(\text{U, V}, \ldots.) = f\,(\text{limit U, limit V}, \ldots.).$$

Hence, we have, in general, the following rule for obtaining the limit of any continuous function of any number of variables.

Substitute for each variable its limit under the law.

It follows, that *those relations which continually exist between variables as they approach their respective limits under a law, will exist between their limits.*

Theorem X. *If unity is the limit of the ratio of any two variables with finite limits, the limit of any function of one will be equal to the limit of the same function of the other.*

Let U and V represent any two variables, giving limit $\left[\frac{\text{U}}{\text{V}}\right] = 1$.

Then,

$$\text{limit}[f\,(\text{U})] = \text{limit}\left[f\left[\frac{\text{U V}}{\text{V}}\right]\right] = f\left(\text{limit}\left[\frac{\text{U}}{\text{V}}\right]\text{limit V}\right) = f(\text{limit V}) = \text{limit}[f(\text{V})].$$

Exercises.

Having limit $\left[\frac{\text{U}}{\text{V}}\right] = 1$, we find,

1. Lim. $(\text{A} \pm \text{U}) = \text{A} \pm \text{lim. U} = \text{A} \pm \text{lim. V}$.

For, $\text{lim. }(\text{A} \pm \text{U}) = \text{lim. }\left[\frac{(\text{A} \pm \text{U})\text{ V}}{\text{V}}\right] = \frac{(\text{A} \pm \text{lim.U})\text{lim.V}}{\text{lim.V}}$

$$= \text{A} \pm \text{lim. }\left[\frac{\text{U}}{\text{V}}\right] \text{ lim.V} = \text{A} \pm \text{lim.V}.$$

2. Lim. $[\text{A U}] = \text{A lim. U} = \text{A lim. V}$.

For, $\text{lim.}[\text{A U}] = \text{lim.}\left[\frac{\text{A U V}}{\text{V}}\right] = \text{A lim.}\left[\frac{\text{U}}{\text{V}}\right] \text{ lim. V} = \text{A lim. V}$.

3. Lim. $\left[\frac{\text{U}}{\text{A}}\right] = \frac{1}{\text{A}} \text{ lim. U} = \frac{1}{\text{A}} \text{ lim. V}$.

For, $\lim.\left[\frac{U}{A}\right]=\lim.\left[\frac{U\,V}{A\,V}\right]=\frac{1}{A}\lim.\left[\frac{U}{V}\right]\lim.\ V=\frac{1}{A}\lim.\ V.$

4. Lim. $U^n=(\lim U)^n=(\lim.\ V)^n$.

For, $\lim.\ U^n=\lim.\left[\frac{U\,V}{V}\right]^n=\lim.\left[\frac{U}{V}\right]^n\lim.\ V^n=(\lim.\ V)^n$.

5. Lim. $\sqrt[n]{U}=\sqrt[n]{\lim.\ U}=\sqrt[n]{\lim.\ V}$.

For, $\lim.\ \sqrt[n]{U}=\lim.\ \sqrt[n]{\frac{UV}{V}}=\lim.\ \sqrt[n]{\frac{U}{V}}\ \lim.\ \sqrt[n]{U}=\sqrt[n]{\lim.\ V}.$

6. Lim. $A^U=A^{\lim.\ U}=A^{\lim.\ V}$.

For, $\lim.\ A^u=\lim.\left[A^{\frac{U}{V}}\right]^V=\left(A^{\lim.\left(\frac{U}{V}\right)}\right)^{\lim.\ V}=A^{\lim.\ V}.$

7. Lim. $\log U=\log\lim.\ U=\log\lim.\ V$.

For, $\lim.\ \log U=\lim.\ (\log.\ U-\log.\ V+\log V)=\lim.\left[\log\frac{U}{V}+\log V\right]$

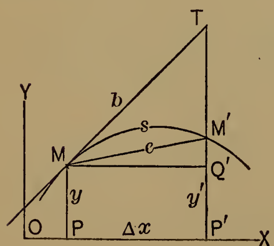
$=\log\lim.\left[\frac{U}{V}\right]+\log.\ \lim.\ V=\log\lim.\ V.$

8. Lim. $\sin U=\sin\lim.\ U=\sin\lim.\ V$.

For, $\lim.\ \sin U=\lim.\ \sin\frac{UV}{V}=\sin\left[\lim.\ \left(\frac{U}{V}\right)\lim.\ V\right]=\sin\lim.\ V.$

Theorem X enables us to substitute either of two variables for the other in any function, without affecting *the limit* of that function, under the law that makes unity the limit of the ratio of the two variables interchanged.

The advantage in so doing arises when we can determine an exact expression for one of the variables and not for the other.



To illustrate, let $MM'=s$ be an arc of a plane curve, PM and $P'M'$ the ordinates of its extremities. Draw the chord MM', and denote PP' by Δx. Under the law that Δx approaches zero, which requires s to approach zero, let it be required to find limit$\left[\frac{\text{arc } MM'}{\Delta x}\right]$.

Having no exact expression for the length of the arc MM', it is impossible to find the limit of the above ratio; but it will be shown hereafter that $\underset{s \to 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\text{chord } MM'}\right]=1.$ Hence, Theorem X,

$$\underset{\Delta x \to 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta x}\right]=\underset{\Delta x \to 0}{\text{limit}}\left[\frac{\text{chord } MM'}{\Delta x}\right]=\underset{\Delta x \to 0}{\text{limit}}\left[\frac{\frac{\Delta x}{\cos Q'MM'}}{\Delta x}\right]$$

$$=\underset{\Delta x \to 0}{\text{limit}}\left[\frac{1}{\cos Q'MM'}\right]=\frac{1}{\cos Q'MT}.$$

It is important to notice that the above substitution is authorized only in taking *the limit* of a function of the arc, for an arc is never equal to its chord.

From Theorem III, we have limit u = limit v, or $\frac{\text{limit u}}{\text{limit v}}=1$, when $u-v=\delta$ is an infinitesimal; and from Theorem VI, we have $\frac{\text{limit u}}{\text{limit v}}=\text{limit}\left[\frac{u}{v}\right]$ when u and v have any finite limit except zero.

Hence, *unity is the limit of the ratio of any two variables with finite limits, not zero, if their difference is an infinitesimal.*

When each of two variables has zero or infinity as a limit it does not follow that the limit of their ratio is unity.

Let u, v, w, and s, be functions of the same variable, giving under a law, limit $\left[\frac{u}{v}\right]=1$, and limit $\left[\frac{w}{s}\right]=1$.

Then will limit $\left[\frac{u}{w}\right]=$ limit $\left[\frac{v}{s}\right]$. For, Theorem X, limit $\left[\frac{u}{w}\right]=$ limit $\left[\frac{v}{w}\right]=$ limit $\left[\frac{v}{s}\right]$.

Applications of the Principles of Limits.

32. $\underset{m \to 0}{\text{Limit}}\left[1+m\right]^{\frac{1}{m}}=\varepsilon.$

Developing $\left(1+m\right)^{\frac{1}{m}}$ by the binomial formula, we have

$$\left(1+m\right)^{\frac{1}{m}}=1+\frac{1}{m}m+\frac{1}{m}\left(\frac{1}{m}-1\right)\frac{m^2}{1.2}+\ \ldots\ldots\ldots\ldots,\ldots\ldots\ldots$$

$$+\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ldots\left(\frac{1}{m}-n+1\right)\frac{m^n}{1.2.\ \ldots\ n}+\ \&c.,$$

which may be written

$$\left(1+m\right)^{\frac{1}{m}}=1+1+\frac{1-m}{1.2}+\frac{(1-m)(1-2m)}{1.2.3}+\dots$$
$$+\frac{(1-m)(1-2m)\dots 1-(n-1)m}{1.2.3.\dots n}+\text{\&c.} \qquad (1)$$

As m approaches zero, each term in (1) approaches the corresponding term of the series

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots+\frac{1}{1.2.3.\dots n}+\text{\&c.} \qquad (2)$$

Hence,

$$\underset{m \ggg\!\to 0}{\text{limit}}\left(1+m\right)^{\frac{1}{m}}=1+1+\frac{1}{1.2.3}+\dots+\frac{1}{1.2.3.\dots n}+\text{\&c.}$$

From Algebra we have

$$\varepsilon=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots+\frac{1}{1.2.3.\dots n}+\text{\&c.}=2.7182818+;$$

in which ε is the base of the Napierian system of logarithms.

Hence, $$\underset{m \ggg\!\to 0}{\text{limit}}\left(1+m\right)^{\frac{1}{m}}=\varepsilon.$$

33. $$\underset{x \ggg\!\to 0}{\text{Limit}}\left[\frac{a^x-1}{x}\right]=\log_e a.$$

In the expression $\frac{x}{a^x-1}$, substitute $1+y$ for a^x, giving

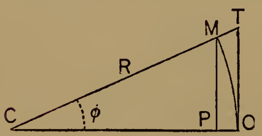
$$\frac{x}{a^x-1}=\frac{\log(1+y)}{y}=\log\left(1+y\right)^{\frac{1}{y}}.$$

Then, since x and y vanish together,

$$\underset{x \ggg\!\to 0}{\text{limit}}\left[\frac{x}{a^x-1}\right]=\underset{y \ggg\!\to 0}{\text{limit}}\left[\log(1+y)^{\frac{1}{y}}\right]=\log\left[\underset{y \ggg\!\to 0}{\text{limit}}\left(1+y\right)^{\frac{1}{y}}\right]=\log\varepsilon=M.$$

Hence, $$\underset{x \ggg\!\to 0}{\text{limit}}\left[\frac{a^x-1}{x}\right]=\frac{1}{\log\varepsilon}=\frac{1}{M}=\log_e a.$$

34. *Unity is the limit of the ratio of an angle to its sin, of an angle to its tan, and of the tan to the sin, as the angle approaches zero.*



Let $OCM=\varphi$ be any angle less than $\frac{\pi}{2}$; then $\tan\varphi>\varphi>\sin\varphi$, and

$$\frac{\tan\varphi}{\sin\varphi}>\frac{\varphi}{\sin\varphi}>1.$$

$$\underset{\varphi \rightarrow 0}{\text{Limit}}\left[\frac{\tan\varphi}{\sin\varphi}\right]=\underset{\varphi \rightarrow 0}{\text{limit}}\left[\frac{1}{\cos\varphi}\right]=1.$$

Hence, $$\underset{\varphi \rightarrow 0}{\text{limit}}\left[\frac{\varphi}{\sin\varphi}\right]=1.$$

Also $$1>\frac{\varphi}{\tan\varphi}>\frac{\sin\varphi}{\tan\varphi}, \text{ and } \underset{\varphi \rightarrow 0}{\text{limit}}\left[\frac{\sin\varphi}{\tan\varphi}\right]=1.$$

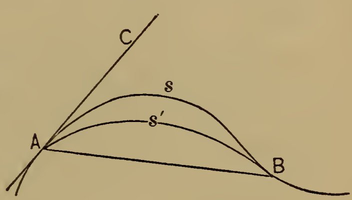
Hence, $$\underset{\varphi \rightarrow 0}{\text{limit}}\left[\frac{\varphi}{\tan\varphi}\right]=1.$$

Let $$u=\sin\varphi, \quad \therefore \quad \varphi=\sin^{-1}u, \text{ and } \underset{u \rightarrow 0}{\text{limit}}\left[\frac{\sin^{-1}u}{u}\right]=1.$$

Let $$u=\tan\varphi, \quad \therefore \quad \varphi=\tan^{-1}u, \text{ and } \underset{u \rightarrow 0}{\text{limit}}\left[\frac{\tan^{-1}u}{u}\right]=1.$$

Similarly, it may be shown that unity is the limit of the ratio of each pair of the lines PM, OT, and OM; as OM approaches zero.

35. Let s be an arc of any curve of double curvature, AC a tangent to it at A, AB the chord corresponding to s; and s' the projection of the curve upon the plane of the tangent AC and chord AB. AC will also be tangent to s' at A. [Des. Geo.].



Assume any number of points upon s, including A and B, and connect adjacent ones by right lines. Represent the chords of s, thus formed, by c, c', etc.

Let θ, θ' etc., denote the ~~cosines of the~~ angles made by the chords respectively with the plane CAB.

The projections of the chords c, c', etc., upon the plane CAB will be chords of s'. [Des. Geo.].

Let the points be taken nearer and nearer to each other, and let n denote the number of chords; then

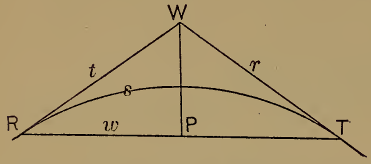
$$s=\underset{n \rightarrow \infty}{\text{limit}}(c+c'+\&c.)=\underset{n \rightarrow \infty}{\text{limit}}\overset{*}{\Sigma}c; \quad \text{and } s'=\underset{n \rightarrow \infty}{\text{limit}}\Sigma c\cos\theta.$$

Under the law $s \rightarrow 0$, the cos of each of the angles θ θ', etc., will approach unity. Hence, § 31, Theorem X,

$$\underset{s \rightarrow 0}{\text{limit}}[s']=\underset{s \rightarrow 0}{\text{limit}}\left[\underset{n \rightarrow \infty}{\text{limit}}\Sigma c\right]; \text{ and } \underset{s \rightarrow 0}{\text{limit}}\left[\frac{s}{s'}\right]=\underset{s \rightarrow 0}{\text{limit}}\left[\frac{\underset{n \rightarrow \infty}{\text{limit}}\Sigma c}{\underset{n \rightarrow \infty}{\text{limit}}\Sigma c}\right]=1.$$

*Σ is used to denote the sum of any number of terms similar in form to the one written after it.

36. *Unity is the limit of the ratio of an arc of any curve to its chord, as the arc approaches zero.*



Let $RT=s$ be an arc of any plane curve, assumed so small that it is concave throughout towards its chord w.

Draw the tangents $RW=t$, and $TW=r$, completing the triangle RWT; and from W draw WP perpendicular to w.

As s approaches zero, we have

$$t+r>s>w, \quad \text{and} \quad w=t\cos R+r\cos T;$$

and since the angles R and T also approach zero,

$$\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{t+r}{w}\right]=\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{t+r}{t\cos R+r\cos T}\right]=1.$$

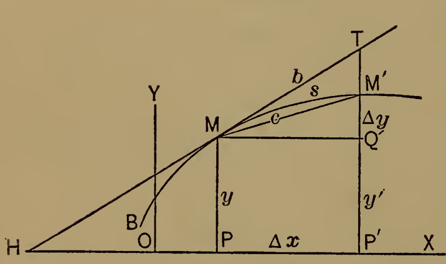
Hence, $$\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{s}{w}\right]=1.$$

In case the given curve is one of double curvature, the tangent RW will not, in general, intersect the tangent at T. Project the curve, and the tangent at T, upon the plane of RT and RW. Let s and TW be their respective projections.

From § 35, $$\underset{\text{arc} \ggg\!\to 0}{\text{limit}}\left[\frac{\text{arc}}{s}\right]=1.$$

Hence, § 31, Theorem X,

$$\underset{\text{arc} \ggg\!\to 0}{\text{limit}}\left[\frac{\text{arc}}{w}\right]=\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{s}{w}\right]=1.$$



37. Let $MM'=s$ be any arc of a plane curve, PM and $P'M'$ the ordinates of its extremities.

Through M draw the chord $MM'=c$, the tangent $MT=b$, and $MQ'=PP'=\Delta x$, parallel to X.

From the triangle $MM'T$, we have $\dfrac{b}{c}=\dfrac{\sin MM'T}{\sin MTM'}$.

As Δx approaches zero, the arc s and the angle $M'MT$ will also approach zero, but the angle T will remain constant. Hence, the angle $MM'T$ will approach $[180° - T]$, and we have

$$\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{b}{c}\right]=\underset{s \ggg\!\to 0}{\text{limit}}\left[\frac{\sin MM'T}{\sin T}\right]=\frac{\sin(180°-T)}{\sin T}=1;$$

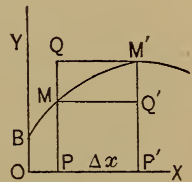
and since, $\quad \frac{b}{c} > \frac{b}{s} > 1,\quad$ we have $\quad \underset{s \ggg\to 0}{\text{limit}} \left[\frac{b}{s}\right] = 1.$

From the same figure we have $\quad b = \frac{\Delta x}{\cos Q'MT}.$

Hence, § 31, Theorem X, and § 36,

$$\underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{s}{\Delta x}\right] = \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{\frac{\Delta x}{\cos Q'MT}}{\Delta x}\right] = \frac{1}{\cos Q'MT} > \text{ or } = 1.$$

38. Let BMM' be any plane curve, and $PMM'P'$ the plane surface included between any arc of the curve, as MM', the ordinates of its extremities, and the axis of X.



Through M and M', respectively, draw MQ' and $M'Q$ parallel to X, and complete the rectangle $MQM'Q'$.

Let $PP' = \Delta x$ approach zero. Then, since

$$PQM'P' > PMM'P' > PMQ'P', \quad \text{and} \quad \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{PQM'P'}{PMQ'P'}\right] = 1,$$

we have $\quad \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{PMM'P'}{PMQ'P'}\right] = 1.$

Hence, § 31, Theorem X,

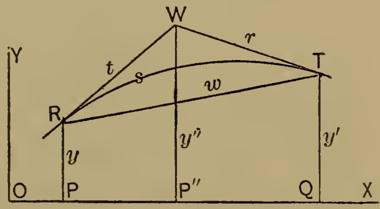
$$\underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{PMM'P'}{\Delta x}\right] = \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{PMQ'P'}{\Delta x}\right] = \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{y\,\Delta x}{\Delta x}\right] = y.$$

If the coördinate axes make an angle θ with each other, then

$$\underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{PMM'P'}{\Delta x}\right] = \underset{\Delta x \ggg\to 0}{\text{limit}} \left[\frac{y \sin\theta\ \Delta x}{\Delta x}\right] = y \sin\theta.$$

39. *Unity is the limit of the ratio of the surface of revolution generated by any plane arc, revolving about an axis in its own plane, to that generated by its chord, as the arc approaches zero.*

Let $RT = s$ be any plane arc in the plane XY, and $PR = y$, and $QT = y'$, the ordinates of its extremities.



Draw the chord $RT = w$, and the tangents $RW = t$, and $TW = r$, forming the triangle RWT. Draw $WP'' = y''$ perpendicular to X.

Let the figure be revolved about X; then

$$\text{sur. gen. by } t = 2\pi\left(\frac{y+y''}{2}\right) t \quad = \quad \pi\,(y+y'')\,t.$$

$$\text{sur. gen. by } r = 2\pi\left(\frac{y''+y'}{2}\right) r \quad = \quad \pi\,(y''+y')\,r.$$

$$\text{sur. gen. by } w = 2\pi\left(\frac{y+y'}{2}\right) w \quad = \quad \pi\,(y+y')w.$$

Under the law that s approaches zero, we have limit y' = limit $y''=y$.

Hence,

$$\lim_{s \to 0} \text{sur. gen. by } (t+r) = \lim_{s \to 0} \pi\,(y+y'')\,t + \lim_{s \to 0} \pi\,(y''+y')\,r = \lim_{s \to 0} [2\,\pi\,y\,(t+r)].$$

$$\lim_{s \to 0} \text{sur. gen. by } w = \lim_{s \to 0} \pi\,(y+y')\,w = \lim_{s \to 0} [2\,\pi\,y\,w].$$

Hence,

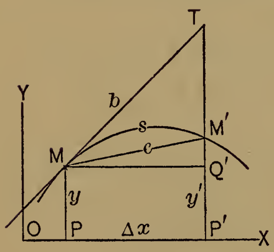
$$\lim_{s \to 0}\left[\frac{\text{sur. gen. by } (t+r)}{\text{sur. gen. by } w}\right] = \lim_{s \to 0}\left[\frac{t+r}{w}\right] = 1, \quad \S\ 36.$$

Since

$$\frac{\text{sur. gen. by } (t+r)}{\text{sur. gen. by } w} > \frac{\text{sur. gen. by } s}{\text{sur. gen. by } w} > 1,$$

$$\lim_{s \to 0}\left[\frac{\text{sur. gen. by } s}{\text{sur. gen. by } w}\right] = 1.$$

Hence, § 31, Theorem X,

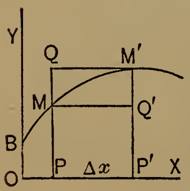


$$\lim_{\Delta x \to 0}\left[\frac{\text{sur. gen. by arc } MM'}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\text{sur. gen. by ch. } MM'}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\pi\,(y+y')\,e}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\pi(y+y')\dfrac{\Delta x}{\cos Q'MM'}}{\Delta x}\right] = \frac{2\,\pi\,y}{\cos Q'MT}.$$



40. Let BMM' be any plane curve, and $PMM'P'$ the plane figure bounded by any arc, as MM', the ordinates of its extremities, and the axis of X. Through M and M', respectively, draw MQ' and $M'Q$ parallel to X, and complete the rectangle $MQM'Q'$.

Let the entire figure be revolved about X, then

vol. gen. by $PQM'P'$ > vol. gen. by $PMM'P'$ > vol. gen. by $PMQ'P'$;

and since

$$\underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{vol. gen. by } PQM'P'}{\text{vol. gen. by } PMQ'P'}\right]=1, \text{ therefore}$$

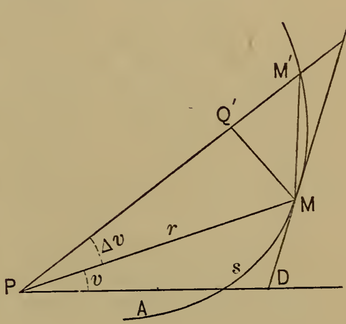
$$\underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{vol. gen. by } PMM'P'}{\text{vol. gen. by } PMQ'P'}\right]=1.$$

Hence, § 31, Theorem X,

$$\underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{vol. gen. by } PMM'P'}{\Delta x}\right]=\underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{vol. gen. by } PMQ'P'}{\Delta x}\right]$$

$$=\underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\pi y^2 \Delta x}{\Delta x}\right]=\pi y^2.$$

41. Let $r=f(v)$ be the polar equation of any plane curve, as AMM', referred to the right line PD, and pole P.



Let $AM=s$ be any portion of the curve, and $PM=r$ the radius vector corresponding to M.

Regarding s as a function of v, § 19, let v be increased by $MPM'=\Delta v$. The arc MM' will be the corresponding increment of s. Draw MQ' perpendicular to PM', and denote PM' by r'. Then, § 31, Theorem X, and § 36, we have

$$\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta v}\right]=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{\text{ch. } MM'}{\Delta v}\right]=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\sqrt{\frac{\overline{MQ'}^2+\overline{Q'M'}^2}{(\Delta v)^2}}$$

$$=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\sqrt{\frac{(r\sin\Delta v)^2+(r'-r\cos\Delta v)^2}{(\Delta v)^2}}$$

$$=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\sqrt{r^2\left(\frac{\sin\Delta v}{\Delta v}\right)^2+\left(\frac{r'-r}{\Delta v}\right)^2}$$

$$=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\sqrt{r^2+\left(\frac{r'-r}{\Delta v}\right)^2}.$$

Also,
$$\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}[\tan Q'M'M]=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{Q'M}{Q'M'}\right]=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{r\sin\Delta v}{r'-r\cos\Delta v}\right]$$

$$=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\left[\frac{r\Delta v}{r'-r}\right]=\tan PMD.$$

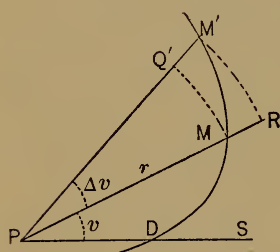
If the radius vector PM coincides with the normal to the curve at M, we have $\quad \text{angle } PMT=PMD=\underset{\Delta v \ggg\!\!\rightarrow 0}{\text{limit}}\ Q'TM=90°.$

Since $MT >$ arc $MM' > MQ'$, and

$$\underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{MT}{MQ'}\right] = \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{1}{\sin Q'TM}\right] = 1,$$

we have $$\underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{arc } MM'}{MQ'}\right] = 1.$$

Hence, $$\underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta v}\right] = \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{MQ'}{\Delta v}\right] = \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{r \sin \Delta v}{\Delta v}\right] = r.$$



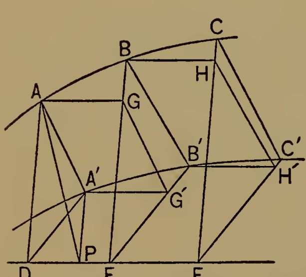
42. Let MPM' be the surface generated by the radius vector $PM = r$, revolving about P, as a pole, from any assumed position, as PM, to any other, as PM'. Let Δv represent the corresponding angle MPM'. With P as a centre, and the radii PM and PM', describe the arcs MQ' and $M'R$ respectively.

Then, since area $RPM' >$ area $MPM' >$ area MPQ', and

$$\underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{area } RPM'}{\text{area } MPQ'}\right] = 1, \quad \text{we have} \quad \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{area } MPM'}{\text{area } MPQ'}\right] = 1.$$

Therefore,

$$\underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{area } MPM'}{\Delta v}\right] = \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\text{area } MPQ'}{\Delta v}\right] = \underset{\Delta v \ggg\!\rightarrow 0}{\text{limit}}\left[\frac{\frac{r^2 \Delta v}{2}}{\Delta v}\right] = \frac{r^2}{2}.$$



43. Let $DABCF$ be a plane figure, and $DA'B'C'F$ its projection on another plane intersecting the first in the right line DF.

Assume any number of points on DF, through which draw right lines parallel to DA and DA' respectively. Through the points in which the first set intersect the curve ABC, and the points in which the second set intersect $A'B'C'$, draw right lines parallel to DF, forming the two sets of parallelograms AE, BF, etc., and $A'E$, $B'F$, etc.

Through AA', the projecting line of A, pass a plane perpendicular to DE cutting the two planes in the right lines AP and PA' respectively.

The angle APA', which we will denote by θ, is that made by the planes with each other.

$A'P=AP\cos\theta$. area $AE=AP\times DE$. area $A'E=A'P\times DE=AP\cos\theta\times DE$
$=$area $AE\cos\theta$.

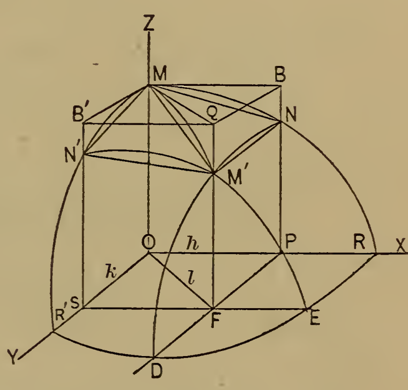
Similarly, each parallelogram in projection is equal to the corresponding one in the given figure into $\cos\theta$.

Let n denote the number of parallelograms, then as the number of points on DF is increased we have

$$DA'B'C'F=\lim_{n\to\infty}(A'E+B'F+\&c.)=\lim_{n\to\infty}(AE+BF+\&c.)\cos\theta$$
$$=DABCF\cos\theta.$$

In a similar manner it may be shown that the area of the projection of any plane figure, is equal to that of the given figure into the cosine of the angle made by its plane with the plane of projection.

44. Let $MNM'N'$ be a portion of any surface, included between the coördinate planes ZX, ZY, and the two planes $N'SE$ and NPD parallel to them respectively.



Let $OP=h$, and $OS=k$.

At M draw the tangents MB and MB' to the curves MN and MN' respectively, and complete the parallelogram $MBQB'$. It will be the portion of the tangent plane to the surface at M, included between the planes which limit $MNM'N'$. Draw the chords MN, NM', $M'N'$, and $N'M$, forming the quadrilateral $MNM'N'$ inscribed in the assumed curved quadrilateral $MNM'N'$. Draw the diagonals MM', MQ and OF.

Conceive the concave surface of the curved triangle MNM' to be entirely covered with inscribed plane triangles, formed by assuming a sufficient number of points, including M, N, and M' on the surface, and connecting those adjacent by right lines.

Let t, t', t'', etc., represent the areas of the triangles respectively; and let θ, θ', θ'', etc., represent the angles made by their respective planes with the plane of the plane triangle MNM'. Then, plane triangle $MNM' = t \cos \theta + t' \cos \theta' + t'' \cos \theta'' + \&c.$;

or, denoting the sum of all the terms in second member by $\Sigma t \cos \theta$, we have plane triangle $MNM' = \Sigma t \cos \theta$.

Denoting the number of the inscribed triangles by n, and increasing them indefinitely, we have

$$\text{curved triangle } MNM' = \lim_{n \to \infty} \Sigma t,$$

$$\text{plane triangle } MNM' = \lim_{n \to \infty} \Sigma t \cos \theta.$$

Suppose h and k to approach 0; or what is equivalent, let OF, represented by l, approach 0. The plane triangle MNM' will approach coincidence with the tangent plane at M. Each of the angles θ, θ', etc., will approach 0, and for each we have

$$\lim_{l \to 0} \left[\frac{1}{\cos}\right] = 1.$$

Hence, § 31, Theorem X,

$$\lim_{l \to 0} \left[\text{plane triangle } MNM'\right] = \lim_{l \to 0} \left[\lim_{n \to \infty} \Sigma t\right],$$

and

$$\lim_{l \to 0} \left[\text{curved triangle } MNM'\right] = \lim_{l \to 0} \left[\lim_{n \to \infty} \Sigma t\right].$$

Therefore,

$$\lim_{l \to 0} \left[\frac{\text{plane tri. } MNM'}{\text{curved tri. } MNM'}\right] = \lim_{l \to 0} \left[\frac{\lim_{n \to \infty} \Sigma t}{\lim_{n \to \infty} \Sigma t}\right] = 1.$$

From § 37,

$$\lim_{l \to 0} \left(\frac{MQ}{MM'}\right) = 1, \quad \lim_{l \to 0} \left(\frac{MB}{MN}\right) = 1, \quad \text{and} \quad \lim_{l \to 0} \left[\frac{\text{angle } BMQ}{\text{angle } NMM'}\right] = 1.$$

Hence,

$$\lim_{l \to 0} \left[\frac{\text{tri. } MBQ}{\text{plane tri. } MNM'}\right] = 1, \quad \text{and} \quad \lim_{l \to 0} \left[\frac{\text{tri. } MBQ}{\text{curved tri. } MNM'}\right] = 1.$$

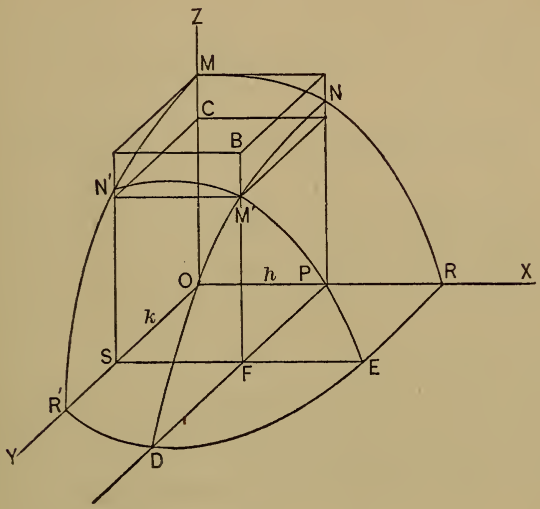
In a similar manner, it may be shown that

$$\lim_{l \to 0} \left[\frac{\text{tri. } MB'Q}{\text{curved tri. } MN'M'}\right] = 1.$$

Hence,

$$\lim_{\substack{h \to 0 \\ k \to 0}} \left[\frac{\text{quadrilateral } MBQB'}{\text{curved quad. } MNM'N'}\right] = 1.$$

45. The volume of $MNM'N' - OF$, included between the coördinate planes, the two planes $N'SE$ and NPD, parallel respectively, to XZ and YZ, and the curved surface $MNM'N'$, is greater than that of the parallelopipedon $OPFS-M'C$, and less than that of $OPFS-MB$.



Let $OP=h$, and $OS=k$, approach zero. Then, since

$$\underset{\substack{h \to 0 \\ k \to 0}}{\text{limit}} \left[\frac{\text{vol. } OPFS-MB}{\text{vol. } OPFS-M'C}\right]=1,$$

we have

$$\underset{\substack{h \to 0 \\ k \to 0}}{\text{limit}} \left[\frac{\text{vol. } MNM'N'-OF}{\text{vol. } OPFS-MB}\right]=1.$$

Hence, § 31, Theorem X, denoting the ordinate MO by z,

$$\underset{\substack{h \to 0 \\ k \to 0}}{\text{limit}} \left[\frac{\text{vol. } MNM'N'-OF}{h\,k}\right]=\underset{\substack{h \to 0 \\ k \to 0}}{\text{limit}} \left[\frac{\text{vol. } OPFS-MB}{h\,k}\right]=\underset{\substack{h \to 0 \\ k \to 0}}{\text{limit}} \left[\frac{z\,h\,k}{h\,k}\right]=z.$$

CHAPTER III.

RATE OF CHANGE OF A FUNCTION.

46. In the function $2x^2$, a change in the variable from 2 to 3, causes the function to change from 8 to 18. If x be again increased the same amount, that is from 3 to 4, the function will increase from 18 to 32. Similarly, with other functions we shall find that, in general, equal changes in the variables do not give equal changes in the corresponding functions.

It is therefore necessary, in referring to a change in a function *corresponding* to a change in the variable, to consider the states from which and to which the function and variable change, as well as the amount of change in each. With that understanding, *corresponding* changes in a function and its variable are mutually dependent.

Thus, having $u=f(x)$ .... (1), hence, § 4, $x=F(u)$ .... (2), increase any value of x in (1) by h, and let k denote the corresponding increment of the function u. Now if the variable u in (2) be increased by k from the state that u in (1) had for the first value of x; the function x in (2) will change by h from and to the same values that the variable x in (1) had.

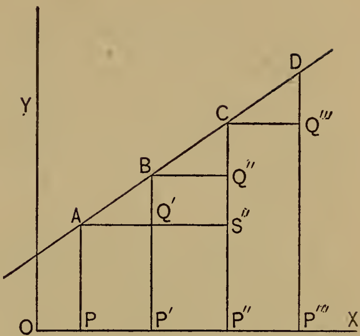
47. A function changes uniformly with respect to a variable, when the ratio of any two increments of the variable is equal to that of the corresponding increments of the function.

It follows that any equal increments of such functions will correspond to equal increments of the variable.

Thus, in $2ax$, let h and l represent any two increments of the variable x. The corresponding increments of the function are $2ah$ and $2al$.

$$\frac{h}{l}=\frac{2ah}{2al}, \text{ and if } h=l, \text{ then } 2ah=2al.$$

Hence, $2ax$ changes uniformly with respect to x.



To illustrate, the function $2ax$ will be represented by the ordinate of some right line, as ABC. Increase any value of x, as OP, by any value, as $PP'=h$. $Q'B$ will be the corresponding increment of the function. Then increase $x=OP$ by any other value, as $PP''=l$, giving the increment $S''C$ to the function. The similar triangles $AQ'B$ and $AS''C$, give $\frac{h}{l}=\frac{Q'B}{S''C}$. Hence, the ordinate of the right line ABC, and therefore the function, changes uniformly with x.

By giving to $x=OP$, any equal increments, as PP', $P'P''$ $P''P'''$, in succession, the corresponding increments of the function, $Q'B$, $Q''C$, and $Q'''D$, are equal to each other.

In a similar manner, it may be shown that any function, which is represented geometrically by the ordinate of a right line, changes uniformly with its variable.

Any function which is of the first degree with respect to the variable, is some particular case of the general form $Ax+B$, in which A and B are constants.

Such functions are represented geometrically by the ordinates of right lines, and will change uniformly with their variables.

48. In the function $2x$;

$x=1$, gives $2x=2$.
$x=2$, gives $2x=4$.
$x=3$, gives $2x=6$.

From which we see that the function increases two units while the variable increases one; in other words, twice as fast.

Having $5x$;

$x=1$, gives $5x=5$.
$x=2$, gives $5x=10$.
$x=3$, gives $5x=15$.

Which shows that the function changes five times faster than the variable.

Hence, different functions, in general, change with their variables with different degrees of rapidity.

The measure of the relative degrees of rapidity of change of a function and its variable at any state, is called the rate of change of the function, with respect to the variable, corresponding to the state.

A rate of change of a function with respect to a variable, corresponding to a state, is an answer to the question: At the state considered, how many times faster than the variable, is the function changing?

49. Since any function, which changes uniformly, receives equal increments for any equal increments of the variable; it follows that the rates of change of such a function, corresponding to different states, must be equal; for otherwise, the function would receive greater or less increments for equal increments of the variable. Hence, the rate of any function, which changes uniformly with respect to a variable, is constant.

50. From the definitions of uniform change and rate, it follows that the rate of a function which changes *uniformly* with respect to a variable, is equal to the ratio of any increment of the function to the corresponding increment of the variable.

Thus, having any $f(x)$, which is of the first degree with respect to x, increase x by any convenient increment h. $f(x+h)-f(x)$ will be the corresponding increment of the function, and the rate will be $\frac{f(x+h)-f(x)}{h}$. This ratio is independent of h, hence h may be made zero without affecting the rate.

Thus, rate of $2x=\frac{2(x+h)-2x}{h}=2.$

Rate of $3x+2=\frac{[3(x+h)+2]-[3x+2]}{h}=3.$

Rate of $5x-3=\frac{[5(x+h)-3]-[5x-3]}{h}=5.$

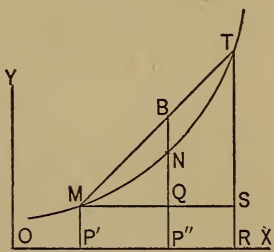
It follows, that the product of the rate of a function, which changes *uniformly*, and any increment of the variable, is the corresponding increment of the function.

51. In the function ax^2, let h and l represent any two increments of x. The corresponding increments of the function are $2axh+ah^2$, and $2axl+al^2$.

The ratio $\frac{h}{l}$ is not, in general, equal to $\frac{2axh+ah^2}{2axl+al^2}$; hence the function ax^2 does not vary uniformly with x.

In a similar manner it may be shown that any function, which is not of the first degree with respect to a variable, does not change uniformly with the variable.

To illustrate, take any function of a degree higher than the first. It will be represented by the ordinate of some curve, as MNT. Increase any value of x, as OP', by $P'P''$, and $P'R$; then, since $\frac{QB}{ST}=\frac{P'P''}{P'R}$, the ratio $\frac{QN}{ST}$, of the corresponding increments of the function is not, in general, equal to $\frac{P'P''}{P'R}$.



52. In the function $2x^2$;

$$\begin{array}{lll} x=1, & \text{gives} & 2x^2=2. \\ & & \quad <6 \\ x=2, & \text{gives} & 2x^2=8. \\ & & \quad <10 \\ x=3, & \text{gives} & 2x^2=18. \\ & & \quad <14 \\ x=4, & \text{gives} & 2x^2=32. \end{array}$$

Which shows that at different states the function $2x^2$ has different rates with respect to x.

Similarly, it may be shown that any function which does not change uniformly has, in general, different rates at different states. In other words, the rate varies with the function and its variable. Any particular rate is, therefore, designated as the rate corresponding to a particular state.

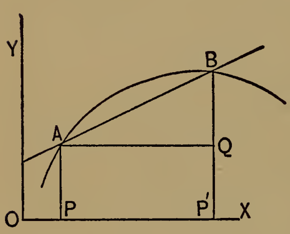
If a function has two or more states corresponding to any value of the variable, each state will have a rate.

If a function has equal states for different values of the variable, it may have a different rate at each; in which case it is necessary to indicate the value of the variable corresponding to the state considered.

53. Let $F(x)$ be any function which does not change uniformly with its variable. Denote its rate, corresponding to any particular state, by R. Increase the corresponding value of x by

h, and let R' represent the rate of the function at the new state $F(x+h)$. Let h be taken so small that the rates between, and including R and R', shall increase or decrease in order. $F(x+h)-F(x)$ will be the corresponding increment of the function.

The ratio $\frac{F(x+h)-F(x)}{h}$ is not the rate R, for the change $F(x+h)-F(x)$ is due to all rates from R to R'; but the ratio $\frac{F(x+h)-F(x)}{h}$ multiplied by h gives the increment $F(x+h)-F(x)$; hence, the ratio $\frac{F(x+h)-F(x)}{h}$ is the rate of *another function of* x, which varying *uniformly* between the states considered, will change by an amount equal to that of the given function,



To illustrate, let the given function be the one represented by the ordinate of the curve AB. Let PA represent the state at which the rate is R; and let $PP'=h$ be the increment of the variable. $P'B$ will then represent the state at which the rate is R', and QB will be the increment of the function corresponding to h.

Draw the right line AB. Its ordinate will represent a function which changes uniformly from the state PA to $P'B$, and by an amount QB equal to that of the given function. Therefore, $QB=F(x+h)-F(x)$, and $\frac{QB}{h}=\frac{F(x+h)-F(x)}{h}$; but $\frac{QB}{h}$ and therefore $\frac{F(x+h)-F(x)}{h}$, is the constant rate of the function represented by the ordinate of the right line AB.

The constant rate of the function represented by the ordinate of the right line must be greater than the least, and less than the greatest rate of the given function for the states under consideration; otherwise the function represented by the ordinate of the right line would change by a less or greater amount than the given function between the states considered. Hence, we have either

$$R<\frac{F(x+h)-F(x)}{h}<R', \quad \text{or} \quad R>\frac{F(x+h)-F(x)}{h}>R';$$

depending upon whether the rates from R to R' are increasing or decreasing.

One or the other of the above relations will exist always as h is diminished numerically; and since, in either case, R is the limit of R' under the law that h approaches zero, we have

$$\underset{h \to 0}{\text{limit}} \left[\frac{F(x+h)-F(x)}{h}\right]=R.$$

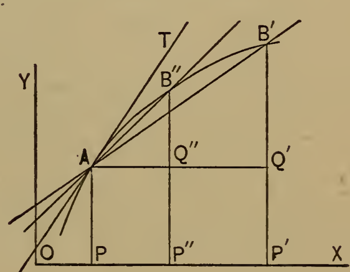
That is, *the rate of change of any function with respect to a variable, corresponding to any state, is equal to the limit of the ratio of any increment of the function, from the state considered, to the corresponding increment of the variable, under the law that the increment of the variable approaches zero.*

The above principle enables us to find the rate of any function with respect to a variable, corresponding to any state, by the following general rule.

Give to the variable any variable increment, and from the corresponding state of the function subtract the primitive. Divide the remainder by the increment of the variable, and determine the limit of this ratio, under the law that the increment of the variable approaches zero. In the result substitute the value of the variable corresponding to the state.

It should be observed, that *a rate*, determined by the above method, is equal to a *limit* of *a ratio* of two infinitesimals, which limit is determin ate ; and that it is not equal to the *ratio* of their limits, which ratio is $\frac{0}{0}$, and therefore indeterminate. See § 31, Theorem VI.

54. To illustrate the changes which occur in *the ratio* of the increment of the function to that of the variable *under the above law;* let the given function be represented by the ordinate of the curve $AB''B'$, and let PA be the state considered.



The ratio, for $h=PP'$, is $\frac{Q'B'}{PP'}$, which is the rate of the function represented by the ordinate of the right line AB'.

For $h = PP''$, *the ratio* $\frac{Q''B''}{PP''}$, is the rate of the function represented by the ordinate of the right line AB''.

As h is diminished, *the ratio is always the rate of a function* represented by the ordinate of a secant approaching the tangent AT; and the limit of *the ratio* is the rate of the function represented by the ordinate of the tangent AT.

That is, the rate of the given function, at the state PA, is the same as that of a uniformly varying function represented by the ordinate of the tangent AT.

This is consistent with previous conceptions and definitions, for the direction of the motion of the point, generating the curve at any position, is along the tangent at the point; and the rate of change of the corresponding ordinate of the curve and tangent, must be the same.

55. § 31, Theorem VI, 3°, shows that the *limit* of the ratio of any increment of a function from any state, to the corresponding increment of the variable, under the above law, is equal to the tangent of the angle made, with the axis of abscissas, by a tangent to the curve, whose ordinate represents the function, at the point corresponding to the state considered. Hence, the rate of a function with respect to a variable at any state, is equal to the tangent of the angle above described.

Exercises.

Find the rate of change of each of the following functions.

1. $2ax$. Ans. $\underset{h \to 0}{\text{limit}} \left[\frac{2a(x+h) - 2ax}{h}\right] = 2a$.

2. x^2. Ans. $\underset{h \to 0}{\text{limit}} \left[\frac{(x+h)^2 - x^2}{h}\right] = 2x$.

3. $ax^2 + bx$. Ans. $\underset{h \to 0}{\text{limit}} \left[\frac{a(x+h)^2 + b(x+h) - (ax^2 + bx)}{h}\right] = 2ax + b$.

4. $\frac{a}{x}$. Ans. $\underset{h \to 0}{\text{limit}} \left[\frac{\frac{a}{x+h} - \frac{a}{x}}{h}\right] = -\frac{a}{x^2}$.

5. $2ax^2$. Ans. $4ax$.

6. x^3. Ans. $3x^2$.

7. $4x^4$. Ans. $16x^3$.

8. $\frac{1}{1+x}$. Ans. $-\frac{1}{(1+x)^2}$.

9. $\frac{2x}{3+x}$. Ans. $\frac{6}{(3+x)^2}$.

10. How is the ordinate of a parabola, corresponding to $x=3$, changing with respect to the abscissa?

$$y=\sqrt{2px}, \quad \therefore \quad \text{rate of } y=\underset{h\ggg\to 0}{\text{limit}}\left[\frac{\sqrt{2p(x+h)}-\sqrt{2px}}{h}\right]$$

$$=\sqrt{2p}\underset{h\ggg\to 0}{\text{limit}}\left[\frac{(x+h)^{\frac{1}{2}}-x^{\frac{1}{2}}}{h}\right]=\sqrt{\frac{p}{2x}},$$

$$\left(\sqrt{\frac{p}{2x}}\right)_{x=3}=\sqrt{\frac{p}{6}} \quad \text{Ans.}$$

11. Same corresponding to focus? Ans. 1.

12. Find the abscissa of the point, of the parabola $y^2=4x$, where the ordinate is changing twice as fast as the abscissa.

$$\text{Rate of } y=2 \quad \therefore \quad 2=\sqrt{\frac{p}{2x}}=\sqrt{\frac{2}{2x}} \quad \therefore \quad x=\frac{1}{4}. \quad \text{Ans.}$$

13. At the vertex of a parabola, how is the ordinate changing as compared with the abscissa?

14. Find the rate of change of the abscissa of a parabola with respect to the ordinate.

Ans. $\frac{y}{p}=\sqrt{\frac{2x}{p}}$.

15. Find the coördinates of the point of the parabola $y^2=8x$, where the abscissa is changing twice as fast as the ordinate.

$$2=\frac{p}{y}=\frac{y}{4}.$$

Ans. $y=8$. $x=8$.

16. Find the rate of change of the ordinate of the right line $2y-3x=12$, with respect to the abscissa.

Ans. $\frac{3}{2}$.

17. A point moves from the origin so that y always increases $\frac{5}{4}$ times as fast as x; find the equation of the line generated.

$\frac{5}{4}=$ tan of angle line makes with X. $\therefore$ Ans. $4y=5x$.

56. Motion.* When a point changes its position with respect to any origin it is said to be in motion with respect to that origin.

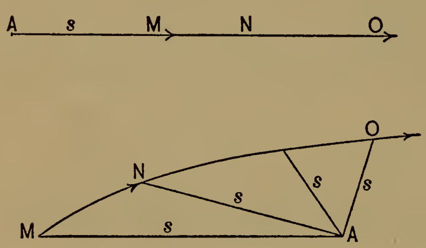
In general, the distance from any origin to a point in motion continually changes, and is a continuous function of the time during which the point moves.

When the distance changes so that *any* two increments of it whatever are proportional to the corresponding intervals of time, the distance changes uniformly with the time. The point is then said to be moving *uniformly*, or with *uniform* motion with respect to the origin.

If the distance does not change uniformly with the time the point is said to be moving with *varied* motion with respect to the origin.

A train of cars moves from a station with varied motion until it attains its greatest speed, after which its motion along the track is uniform while it maintains that speed.

With uniform motion equal distances are passed over in any equal portions of time, and with varied motion unequal distances are passed over in equal portions of time.



Let s represent the variable distance from any origin as A, to a point moving on any line, as MNO; and let t denote the number of units of time during which the point moves; then $s=f(t)$.

If $f(t)$ is of the first degree with respect to t, the distance s will change uniformly; otherwise the point approaches, or recedes from the origin with varied motion, § 47, § 51.

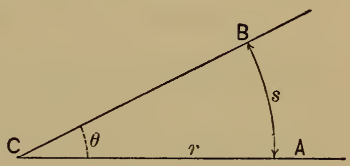
The rate of change of s, regarded as a function of t, corresponding to any position of the moving point, is called the *rate of motion* of the moving point with respect to the origin; and since

*Motion, without regard to cause, is generally discussed under the head of Kinematics, but many important applications of the Calculus involve motion, therefore, some of the definitions and principles of Kinematics are here and elsewhere introduced.

uniform motion causes s to change uniformly with t, the rate of motion, in such cases, is constant. § 49.

In varied motion, the rate varies with t, and is therefore a function of t.

Let C be a fixed point, CA a fixed right line, and B a point in motion so that the angle ACB, denoted by θ, is changing. Then the line CB is said to have an *angular motion* with respect to, or about, C.



Let s represent the length of the varying arc, of any convenient circle, subtending θ, giving $\theta = \frac{s}{r}$.

Both θ and s are functions of the time during which CB moves.

Angular motion is *uniform* when any two increments of the angle, or arc subtending the angle, are proportional to the corresponding intervals of time; otherwise it is *varied*.

57. A function of two variables changes uniformly with respect to both variables when it receives equal increments corresponding to any equal increments of each variable.

Every function of two variables, which is of the first degree with respect to the variables, must be some particular case of the general form $Ax+By+C$, in which A, B and C are constants.

Placing $z=Ax+By+C$, and increasing x by h, and y by k, we have for a second state $z'=A(x+h)+B(y+k)+C$.

Again increasing x by h, and y by k, we have for a third state $z''=A(x+2h)+B(y+2k)+C$.

$z'-z=Ah+Bk$, is the increment of the function from the primitive to the second state.

$z''-z'=Ah+Bk$, is the increment from the second to the third state.

These increments of the function are equal, and correspond to any equal increments of each variable. Hence, any function of two variables, which is of the first degree with respect to the variables, changes uniformly with respect to both variables.

58. Let $z=f(x, y)=Ax^2+By+C$. Increase the variables, respectively, by h and k, giving the new states,

$$z'=A(x+h)^2+B(y+k)+C, \quad \text{and} \quad z''=A(x+2h)^2+B(y+2k)+C.$$

Hence,

$$z'-z=2Axh+Ah^2+Bk, \quad \text{and} \quad z''-z'=2Axh+3Ah^2+Bk.$$

The increments of the function, corresponding to equal increments of each variable, are unequal, hence the function does not change uniformly with respect to both variables.

In a similar manner, it may be shown that any function of two variables, which is not of the first degree with respect to the variables, does not change uniformly with respect to both variables.

59. Any function of two variables which changes uniformly with respect to both variables must be of the first degree with respect to the variables, and its form must be some particular case of the general expression, $Ax+By+C$.

It also follows, that the surface, whose ordinate represents a function of two variables which changes uniformly with both variables, is a plane.

60. In a similar manner, it may be shown that any function of any number of variables, which changes uniformly with respect to all the variables, must be of the first degree with respect to the variables.

61. The Calculus is that branch of mathematics by which measurements, relations, and properties of functions are determined from their rates of change.

It is generally divided into two parts.

Part I, called Differential Calculus, embraces the deductions and uses of the rates of functions.

Part II, called Integral Calculus, treats primarily of methods for determining functions from their rates.

CHAPTER IV.

THE DIFFERENTIAL AND DIFFERENTIAL COEFFICIENT OF A FUNCTION.

62. An arbitrary amount of change assumed for the independent variable is called *the differential of the variable.*

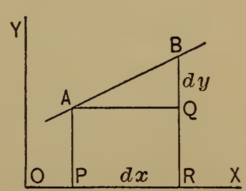
It is represented by writing the letter d before the symbol for the variable; thus dx, read "differential of x," denotes the differential of x.

It is always assumed as positive, and remains constant throughout the same discussion unless otherwise stated.

63. The differential of a function of a single variable is *the change that the function would undergo from any state, were it to retain its rate at that state, while the variable changed by its differential.*

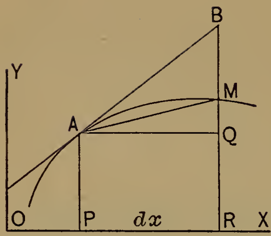
The differential of a function which varies uniformly with its variable, is the change in the function corresponding to that assumed for the variable.

To illustrate, let PA be any state of the uniformly varying function represented by the ordinate of the right line AB. Assume $PR = dx$.



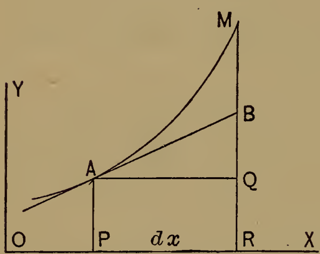
QB, the corresponding change in the function, is the differential of the function.

The differential of a function which does not vary uniformly with its variable, is not, in general, the corresponding change in the function; but it is the corresponding change of a function having a constant rate equal to that of the given function at the state considered: or, in other words, it is the change that the function would undergo, were it to continue to change from any state, *as it is changing at that state, uniformly* with a change in the variable equal to its differential.



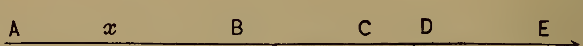
To illustrate, let PA be any state of a given function represented by the ordinate of the curve AM. Assume $PR = dx$.

QM is the corresponding change in the function; but QB, the corresponding change in the function represented by the ordinate of the right line AB drawn tangent to AM at A, is the differential of the given function corresponding to the state PA. For the function represented by the ordinate of AB has a constant rate equal to that of the given function at PA, § 54; and QB is the change that the given function would undergo; were it to continue to change from the state PA, *as it is changing at that state*, *uniformly* with a change in x equal to dx.



The differential of a function which does not vary uniformly with its variable, may be less than the corresponding change in the function. Thus, $QB, < QM$, is the differential of the function represented by the ordinate of the curve AM, corresponding to PA.

A train of cars in motion affords a familiar example of a differential of a function.



Suppose that a train of cars starts from the station A, and moves in the direction AE with a continuously increasing speed. Let x denote the variable distance of the train from A at any instant; it will be a function of the time, represented by t, during which the train has moved, giving $x = f(t)$.

Suppose the train to have arrived at B, for which point $x = AB$. Let BD represent the distance that the train will actually run in the next unit of time, say one second, with its rate constantly increasing.

Let BC represent the distance that the train would run, if it were to move from B with its rate at that point unchanged, in a second. Then will the distance BC represent the differential of x regarded as a function of t, corresponding to the state $x=AB$; and one second will be the differential of the variable.

The differential of a function is denoted by writing the letter d before the function or its symbol.

Thus, $d\,2ax^3$, read "differential of $2ax^3$," indicates the differential of the function $2ax^3$.

Having $y=\log\sqrt{ax^2}$, we write $dy=d\log\sqrt{ax^2}$.

$\frac{dy}{dx}dx$ denotes the differential of y regarded as a function of x;

and $\frac{dx}{dy}dy$ is a symbol for the differential of the inverse function; that is, of x regarded as a function of y.

64. From the definition of a differential of a function, and from § 50, it follows, that a differential of a function is the product of two factors; one of which is the rate of change of the function at the state considered, and the other is the assumed differential of the variable. Hence, the differential of any given function may be determined by finding its rate, by the general rule, § 53, and multiplying it by the differential of the variable. Thus, having the function $2x^2$, we find, § 53,

$$\operatorname*{limit}_{h \ggg\!\to 0}\left[\frac{2(x+h)^2-2x^2}{h}\right]=4x=\text{ rate corresponding to any state.}$$

$4xdx$ is, therefore, a general expression for the differential of $2x^2$, and is written $d\,2x^2=4xdx$.

Its value corresponding to any particular state is obtained by substituting the value of the variable corresponding to the state; thus, for $x=2$, we have $d2x^2=8dx$.

65. Since the *rate of change* of a function is the coefficient of the differential of the variable, in the expression for the differential of the function; writers on the Calculus have, in general, adopted for it the name "*differential coefficient.*"

The *differential of a function* is therefore equal to the product of the *differential coefficient* by the *differential of the variable.*

It follows, that the differential coefficient is the quotient of the differential of the function by the differential of the variable. Thus, having $d\,2x^2 = 4x\,dx$, the differential coefficient is $\frac{d2x^2}{dx} = 4x$; or, having denoted any function of x, by y, and its differential by dy, its differential coefficient is represented by $\frac{dy}{dx}$.

The differential coefficient of any function of a single variable may be determined by the general rule, § 53.

Thus, having $y = f(x)$, in which y represents any function, of any variable x, let y' denote the new state of the function corresponding to the increment h of the variable. Then,

$$\underset{h \ggg\!\to 0}{\text{limit}} \left[\frac{f(x+h)-f(x)}{h}\right] = \underset{h \ggg\!\to 0}{\text{limit}} \left[\frac{y'-y}{h}\right] = \frac{dy}{dx};$$

or, representing the increment of x by Δx, and that of y by Δy, we have

$$\underset{h \ggg\!\to 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}.$$

Since the increment of the variable, represented by h or Δx, varies, it may happen that $h = \Delta x = dx$. It is exceedingly important to observe, however, that the corresponding value of $y'-y$ or Δy, *is not*, in general, equal to dy; for that would give

$$\left(\frac{y'-y}{h}\right)_{h=dx} = \left(\frac{\Delta y}{\Delta x}\right)_{\Delta x = dx} = \frac{dy}{dx};$$

which, in general, is impossible, since $\frac{dy}{dx}$ is not a value of the ratio $\frac{y'-y}{h}$, but is its limit under the law that h vanishes.

If, however, the function changes uniformly with respect to the variable, $\frac{y'-y}{h}$ will be constant for all values of h, § 50; and $y'-y$ will be equal to dy when h is equal to dx.

66. The following important facts in regard to a differential coefficient should now suggest themselves to the student.

It is zero for a constant quantity. In other words, a constant has no differential coefficient.

It is constant for any function which varies uniformly.

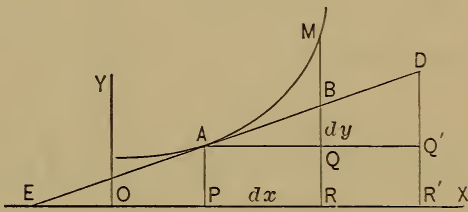
It varies from state to state for any function which does not vary uniformly.

In general, therefore, it is a function of the variable.

It may have values from $-\infty$ to $+\infty$.

Having represented a function by the ordinate of a curve, the differential coefficient is equal to the tangent of the angle made with the axis of abscissas, by a tangent to the curve at the point corresponding to the state considered, § 55.

Thus, assuming $PR = dx$, the differential coefficient of the function, represented by the ordinate of the curve AM, at the state PA, is equal to

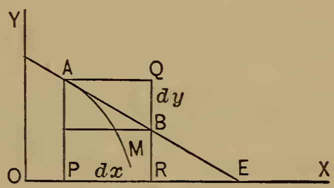


$$\tan XEA = \tan QAB = \frac{dy}{dx}.$$

It should be noticed, that $\frac{dy}{dx}$ is independent of the value assumed for the differential of the variable; for if $PR' = dx$, then $Q'D = dy$, and we have, as before, $\frac{dy}{dx} = tan XEA$.

In this illustration the function is an increasing one, and its differential coefficient is positive, since it is equal to the tangent of an acute angle.

In case the function represented by the ordinate of AM, is a decreasing one, its differential coefficient corresponding to PA is negative, since the angle XEA is then obtuse.



67. The following facts should now be apparent concerning a differential of a function.

It is zero for a constant.

It is constant for any function which varies uniformly.

It is a function of the variable for any function which does not vary uniformly.

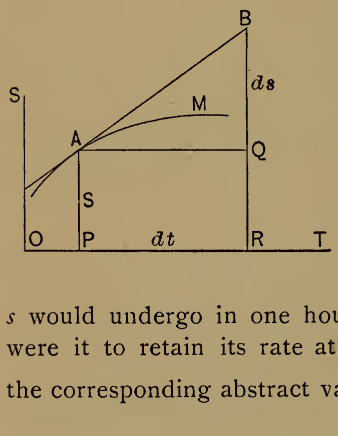
Its value depends upon that of the differential coefficient, and that assumed for the differential of the variable.

It may have values from $-\infty$ to $+\infty$.

It will be numerically greater or less than the differential coefficient depending upon whether the differential of the variable is assumed greater or less than unity.

It has the same sign as its differential coefficient.

68. If the differential of the variable is assumed equal to the unit of the variable, the differential of a function and the corresponding differential coefficient, will have the same numerical value. Thus, if $\frac{dy}{dx}=2$, and $dx=1$ inch, we have, $\frac{dy}{dx}dx=2$ inches. In such cases the differential of the function expresses the rate in terms of the unit of the variable; and since it is more definite, it is frequently used instead of the differential coefficient.



To illustrate, let s denote any variable distance regarded as a function of time, giving $s=f(t)$. Assuming any convenient length to represent the unit of t, we may, by substituting s for y and t for x, § 20, determine a line, as AM, whose ordinate represents the given function.

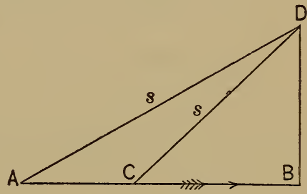
If $PR=dt$ represents one hour, $\frac{ds}{dt}\,dt=QB$ represents the change that s would undergo in one hour, from the state represented by PA, were it to retain its rate at that state; and is more definite than the corresponding abstract value of $\frac{ds}{dt}$.

69. The differential coefficient of the variable distance from any origin to a point in motion, regarded as a function of the time of the motion, is called the **velocity** of the moving point with respect to that origin.

Representing the variable distance by s, the symbol for the velocity is $\frac{ds}{dt}$.

For the reasons described, velocity *is measured* by the product of $\frac{ds}{dt}$ and the distance assumed to represent the unit of time.

That is, *the measure* of the velocity of a point in motion at any instant, in any required direction, *is the distance in that direction, that the point would go in the next unit of time, were it to retain its rate at that instant.*



It should be noticed that the distance referred to above, and represented by s, may, or may not, be estimated along the line or path upon which the body moves. Thus, if a point moves from A towards B, and the velocity at any point, as C, in the direction AB is required, the distance s is estimated along the path described; but if the rate or velocity with which a point, moving from A to B, is approaching D is required, s must represent the variable distance from the moving point to D, in order that $\frac{ds}{dt}$ shall be the required rate of motion.

Since velocity is a rate, it is constant in uniform motion, and a variable function of time in varied motion. § 56.

The differential coefficient of velocity regarded as a function of time is called **acceleration**. It is denoted by $\frac{dv}{dt}$, in which, v represents velocity.

Acceleration is generally expressed in terms of the distance which represents the unit of time.

The differential coefficient of any varying angle regarded as a function of the time is called **angular velocity.**

Representing any varying angle by θ, and its angular velocity by ω, we have $\omega = \frac{d\theta}{dt}$.

If s denotes the varying arc, of a circle whose radius is r, which subtends θ, we have

$$\theta = \frac{s}{r}; \quad \text{hence,} \quad \omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt}*.$$

That is, angular velocity is equal to the actual velocity of a point, describing any convenient circle about the vertex of the angle as a centre, divided by its radius.

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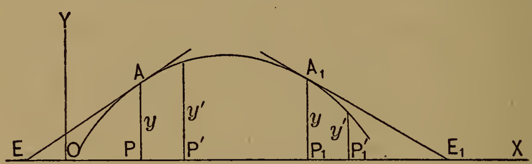
*Assume this result for the present.

It is customary in applied mathematics to consider the radius equal to the unit of distance used in any particular case. Angular velocity will then be measured by the actual velocity of a point at *the* unit's distance from the vertex.

The differential coefficient of angular velocity regarded as a function of time is called **angular acceleration.**

It is denoted by $\frac{d\omega}{dt}$, in which ω represents angular velocity.

70. Let $y=PA$ represent any state of any increasing function of x; and y' the new state corresponding to an increment $PP'=h$



of the variable. $\frac{y'-y}{h}$ will be positive, provided h is assumed sufficiently small, and will remain so as h approaches zero, § 14.

Hence, § 31, Theorem I, $\lim_{h \to 0}\left[\frac{y'-y}{h}\right]=\frac{dy}{dx}=\tan XEA$, is positive.

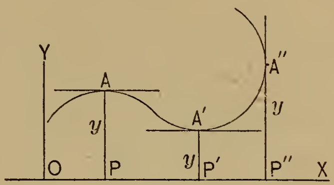
That is, *the differential coefficient corresponding to any state of an increasing function is positive.*

Let $y=P_1A_1$ represent any state of a decreasing function; and y' its new state due to an increment of the variable equal to $P_1P_1'=h$. Then $\frac{y'-y}{h}$ will be negative, if h is small enough, and will remain so as h approaches zero.

Hence, $\lim_{h \to 0}\left[\frac{y'-y}{h}\right]=\frac{dy}{dx}=\tan XE_1A_1$, is negative.

That is, *the differential coefficient corresponding to any state of a decreasing function is negative.*

It follows, that *a function is increasing when its differential coefficient is positive, and decreasing when it is negative.*



If for any value of the variable the differential coefficient is zero, the function is neither increasing nor decreasing, and the tangents at the corresponding points of the line whose ordinate represents the function, are parallel to the axis of X.

If the differential coefficient is infinity, the rate of the function is infinite; and the tangents at the corresponding points of the line whose ordinate represents the function, are perpendicular to the axis of X.

71. Let $\varphi(r)$ and $\psi(x)$ represent any two functions of the same variable which are equal in all their successive states, giving $\varphi(x)=\psi(x)$ . . . . . (1). Increase x by Δx, and we have $\varphi(x+\Delta x)=\psi(x+\Delta x)$ . . . . . (2).

Subtract (1) from (2), member from member; divide both members of the resulting equation by Δx, and we have

$$\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x}=\frac{\psi(x+\Delta x)-\psi(x)}{\Delta x}, \text{ for all values of } x \text{ and } \Delta x.$$

Hence, $$\underset{\Delta x \ggg 0}{\text{limit}}\left[\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x}\right]=\underset{\Delta x \ggg 0}{\text{limit}}\left[\frac{\psi(x+\Delta x)-\psi(x)}{\Delta x}\right],$$

§ 31, Theorem II. Therefore,

$$\frac{d\varphi(x)}{dx}=\frac{d\psi(x)}{dx}; \quad \text{also} \quad d\varphi(x)=d\psi(x).$$

That is, *if two functions of the same variable are equal in all their successive states, their corresponding differentials are equal.*

Cor. *If any two corresponding states of two differentials of functions of the same variable, are unequal, the functions are not equal in all their successive states.*

72. Having given $f(x)\pm C$, in which C represents any constant, we have, by the application of the general rule § 53,

$$\underset{h \ggg 0}{\text{limit}}\left[\frac{[f(x+h)\pm C]-[f(x)\pm C]}{h}\right]=\underset{h \ggg 0}{\text{limit}}\left[\frac{f(x+h)-f(x)}{h}\right].$$

Hence, $$\frac{d(f(x)\pm C)}{dx}=\frac{df(x)}{dx}; \quad \text{and} \quad d(f(x)\pm C)=df(x).$$

That is, *the differential of a function plus or minus a constant is equal to the differential of the function.*

Cor. *If two corresponding differentials are equal, it does not follow that the functions from which they were derived are equal.*

73. Let $y=f(x)$ . . . (1), and $x=F(y)$ . . . (2), be direct and inverse functions.

In (2) increase *any* value of y by k, and denote the corresponding increment $x'-x$, of x by h.

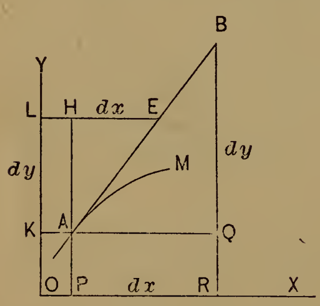
In (1) give the increment h to that value of x, which in (2) corresponded to the value of y that was increased by k, then y in (1) will receive an increment $y'-y$ equal to, and corresponding to k, the assumed increment of y in (2). § 46.

Hence, $$\frac{y'-y}{h}=\frac{k}{x'-x}=\frac{1}{\frac{x'-x}{k}}.$$

Taking their limits under the law $h \ggg\!\!\rightarrow 0$, which requires $k \ggg\!\!\rightarrow 0$, we have $$\underset{h \ggg\rightarrow 0}{\text{limit}}\left[\frac{y'-y}{h}\right]=\frac{1}{\underset{k \ggg\rightarrow 0}{\text{limit}}\left[\frac{x'-x}{k}\right]}.$$

Hence, $$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}.$$

That is, *corresponding to any value of x, the differential coefficient of y regarded as a function of x, is the reciprocal of the differential coefficient of the inverse function.*



To illustrate, let the function y be represented by the ordinate of the curve AM. Assume $dx=PR$, and from the figure we have, corresponding to the state PA,

$$\frac{QB}{PR}=\frac{dy}{dx}=\tan QAB.$$

The inverse function will be represented by the abscissa of the curve AM regarded as a function of the ordinate, and assuming $dy=KL$, we have for the state KA, corresponding to A,

$$\frac{HE}{AH}=\frac{dx}{dy}=\tan EAH. \qquad EAH=90^\circ-QAB.$$

Hence,

$$\tan QAB = \cot EAH = \frac{1}{\tan EAH}; \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

It should be observed that, in general, dy in the first member of the above equation is not the same as dy in the second; for the first is the differential of y as a function, which, in general, is a variable; and the second is a differential of y as the independent variable. The same remarks apply to dx, in the two members, taken in reverse order.

The figure illustrates the differences referred to.

74. Let y be an implicit function of x, the relation being given by the two equations

$y = f(u)$ . . . . (1). $u = \varphi(x)$ . . . . (2).

Increase *any* value of x by h, and denote the corresponding increment $u' - u$, of u by k.

In (1), increase by k that value of u which in (2) corresponds to the value of x that was increased by h, then y in (1) will receive its corresponding increment $y' - y$.

Hence, since $u' - u = k$, and the increments of x, u, and y correspond to the same value of x, we have

$$\frac{y'-y}{h} = \frac{y'-y}{k} \times \frac{u'-u}{h}.$$

Taking their limits, under the law $h \ggg\!\!\to 0$, which requires $k \ggg\!\!\to 0$, we have

$$\underset{h \ggg\to 0}{\text{limit}} \left[\frac{y'-y}{h}\right] = \underset{k \ggg\to 0}{\text{limit}} \left[\frac{y'-y}{k}\right] \times \underset{h \ggg\to 0}{\text{limit}} \left[\frac{u'-u}{h}\right].$$

Hence,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{dy}}{\mathbf{du}} \times \frac{\mathbf{du}}{\mathbf{dx}}.$$

That is, *corresponding to any value of x, the differential coefficient of y regarded as a function of x, is equal to the product of the differential coefficient of y regarded as a function of u, by the differential coefficient of u regarded as a function of x.*

Similarly, having $y = f(u)$, $u = \varphi(x)$, $x = \psi(s)$, we find

$$\frac{dy}{ds} = \frac{dy}{du} \times \frac{du}{dx} \times \frac{dx}{ds};$$

and the same form holds true whatever be the number of the intermediate functions.

If we have $y=f(u)$ . . . (1), and $x=\psi(u)$ . . . (2); (2) may be written $u=\varphi(x)$ . . . (3).

Hence, from (1) and (3), $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$.

But, § 73, $\dfrac{du}{dx}=\dfrac{1}{\dfrac{dx}{du}}$. Hence, $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{du}}{\dfrac{dx}{du}}$.

That is, *corresponding to any value of x, the differential coefficient of y regarded as a function of x, is equal to the quotient of the differential coefficient of y regarded as a function of u, by the differential coefficient of x regarded as a function of u.*

Examples.

Given

1. $y=au^2$, $u=bx$. . . . . $\dfrac{dy}{dx}=2ab^2x$.

2. $z=ay^2$, $y^2=2px$. . . . . . . $\dfrac{dz}{dx}=2ap$.

3. $y=f(u)$, $x=\varphi(u)$, $x=\psi(s)$. . . . . . $\dfrac{dy}{ds}=\dfrac{\dfrac{dy}{du}}{\dfrac{dx}{du}}\times\dfrac{dx}{ds}$.

4. $y=u^2$, $x=3u$, $x=2s^2$. . . . . . . $\dfrac{dy}{ds}=\dfrac{16s^2}{9}$.

5. $y=f(u)$, $u=F(s)$, $z=\psi(s)$. . . . . . $\dfrac{dy}{dz}=\dfrac{dy}{du}\times\dfrac{\dfrac{du}{ds}}{\dfrac{dz}{ds}}$.

6. $y=f(u)$, $v=\varphi(u)$, $v=\psi(s)$, $z=F(s)$, $z=F_2(x)$. $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{du}}{\dfrac{dv}{du}}\times\dfrac{\dfrac{dv}{ds}}{\dfrac{dz}{ds}}\times\dfrac{dz}{dx}$.

CHAPTER V.

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE.

75. The differential of any function of a single variable may always be determined by applying the general rule, § 53, and multiplying the result by the differential of the variable.

By applying the general rule, § 53, to a general representative of any particular kind of function*, there will result a *particular form*, or *rule, for differentiating* such functions, which is generally used in practice.

76. Differential of the Product of a Function and a Constant. Let $Cf(x)$ represent the product of any function by any constant denoted by C.

Applying the general rule, § 53, we have

$$\lim_{h \to 0}\left[\frac{Cf(x+h)-Cf(x)}{h}\right]=C\lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=C\frac{df(x)}{dx}.$$

Hence, $\dfrac{dCf(x)}{dx}=\dfrac{Cdf(x)}{dx}$; and $dCf(x)=Cdf(x)$.

That is, *the differential of the product of a function and a constant is equal to the product of the constant and the differential of the function.*

Cor. *The differential of the quotient of a function by a constant is equal to the quotient of the differential of the function by the constant.*

77. Differential of the Sum or Difference of any Number of Functions. Let $y=u\pm s\pm t\pm$etc, in which, u, s, t, etc. are any functions of any variable, as x.

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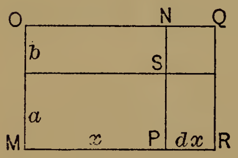
*Functions of a single variable only are considered in this chapter.

Applying the general rule, § 53, we have, § 31, Theorem IV,

$$\lim_{h \to 0}\left[\frac{y'-y}{h}\right]=\lim_{h \to 0}\left[\frac{u'-u}{h}\right]\pm\lim_{h \to 0}\left[\frac{s'-s}{h}\right]\pm\lim_{h \to 0}\left[\frac{t-t'}{h}\right]\pm\text{etc.}$$

Hence, $\dfrac{dy}{dx}=\dfrac{du}{dx}\pm\dfrac{ds}{dx}\pm\dfrac{dt}{dx}\pm$ etc., and $dy=du\pm ds\pm dt\pm$ etc.

That is, *the differential of the sum or difference of any number of functions of the same variable is equal to the sum or difference of their differentials.*



To illustrate, let the side $PM=x$, of the rect. MN be variable, and the side $MO=a+b$, constant. The rect. MN will be equal to the sum of the two rects., MS and OS, giving, rect. $MN=ax+bx$. Increase x by $PR=dx$; then from the definition of a differential, § 63, we have

$$d\,MN=PQ=adx+bdx.$$

EXAMPLES.

$d(x^2-2x)=2(x-1)dx.$ $\qquad$ $d(x^2-3ax+cx^2)=(2x-3a+2cx)dx.$

$d\left(\dfrac{x}{a}+bx^2\right)=\left(\dfrac{1}{a}+2bx\right)dx.$ $\qquad$ $d\left[(a+x)-(x-b)+cx^2-\dfrac{x}{3}\right]=\left(2cx-\dfrac{1}{3}\right)dx.$

78. Differential of the Product of any Number of Functions. Let yz be the product of any two functions of any variable, as x.

Applying the general rule, § 53, we have

$$\lim_{\Delta x \to 0}\left[\frac{y'z'-yz}{\Delta x}\right]=\lim_{\Delta x \to 0}\left[\frac{z\Delta y+y\Delta z+\Delta y\Delta z}{\Delta x}\right]$$
$$=\lim_{\Delta x \to 0}\left[z\frac{\Delta y}{\Delta x}+(y+\Delta y)\frac{\Delta z}{\Delta x}\right],$$

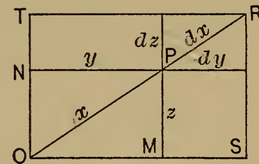
Hence, § 31, Theorems IV and V,

$$\lim_{\Delta x \to 0}\left[\frac{y'z'-yz}{\Delta x}\right]=z\lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right]+\lim_{\Delta y \to 0}[y+\Delta y]\lim_{\Delta x \to 0}\left[\frac{\Delta z}{\Delta x}\right].$$

Therefore, $\dfrac{dyz}{dx}=z\dfrac{dy}{dx}+y\dfrac{dz}{dx}$; and $dyz=zdy+ydz.$

That is, *the differential of the product of any two functions of the same variable is equal to the sum of the products of each function and the differential of the other.*

To illustrate geometrically, let $ONPM$ be a state of a rectangle with a variable diagonal represented by x. Two adjacent sides, denoted by z and y respectively, will be functions of x, and yz will be the variable area of the rectangle. Assume $dx=PR$, and complete the rects. $PT=ydz$, and $PS=zdy$. Then, since $dyz=ydz+zdy$, we have d (rect. OP) = rect. $PT+$ rect. PS.



A consideration of the figure and the law of change shows that the sum of the two rects. PT and PS is the amount of change required by the definition of a differential. § 63.

Let vsu be the product of any three functions of the same variable. Place $vs=r$, giving $vsu=ru$.

Differentiating, we have $dvsu=dru=rdu+udr$, in which,

$dr=vds+sdv$. Hence, by substitution,

$$dvsu=vsdu+vuds+sudv \quad . \quad . \quad . \quad (1)$$

Having the product of four functions of the same variable, $vsuw$, we may place $vsu=r$, and in a manner similar to above, deduce

$$dvsuw=vsudw+vswdu+vuwds+suwdv \quad . \quad . \quad . \quad (2)$$

In the same way, it may be shown that *the differential of the product of any number of functions of the same variable is equal to the sum of the products of the differential of each function and all the others.*

79. Dividing each member of Eq. (2) by $vsuw$, we have

$$\frac{dvsuw}{vsuw}=\frac{dw}{w}+\frac{du}{u}+\frac{ds}{s}+\frac{dv}{v}.$$

Similarly, it may be shown that *the differential of the product of any number of functions of the same variable, divided by their product, is equal to the sum of the quotients of the differential of each function by the function itself.*

80. Differential of a Fraction. Let $\frac{u}{s}$ be any fraction, in which u and s are functions of the same variable.

Place $\frac{u}{s} = y$, then $u = sy$; and, § 78, $du = yds + sdy$.

Hence, $sdy = du - yds$, or $sdy = du - \frac{u}{s} ds = \frac{sdu - uds}{s}$.

Therefore, $$dy = d\frac{u}{s} = \frac{sdu - uds}{s^2}.$$

Hence, *the differential of a fraction is equal to the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

If the numerator is a constant denoted by C, we have

$$d\frac{C}{s} = -\frac{Cds}{s^2}.$$

If the denominator is a constant denoted by C, we have

$$d\frac{u}{C} = \frac{du}{C}.$$

81. Differential of y^m. Let y represent any function of any variable, and m any constant.

1°. If m is entire and positive, $y^m = yyy \ldots$; and § 78, $dy^m = y^{m-1}dy + y^{m-1}dy + \text{etc.} = my^{m-1}dy$.

2°. If m is a positive fraction, equal to $\frac{p}{q}$; p and q being entire and positive, we have

$$y^m = y^{\frac{p}{q}}, \quad \text{and} \quad (y^m)^q = y^p. \quad \text{Hence, 1°.}$$

$$d(y^m)^q = q(y^m)^{q-1} d(y^m) = py^{p-1}dy.$$

Whence,

$$d(y^m) = \frac{p\,y^{p-1}\,dy}{q\,(y^m)^{q-1}} = \frac{p\,y^{p-1}\,y^m}{q\,(y^m)^q}dy = \frac{p\,y^{p-1}\,y^{\frac{p}{q}}}{q\,y^p}\,dy = \frac{p}{q}y^{\frac{p}{q}-1}\,dy = m\,y^{m-1}\,dy.$$

3°. If m is negative, represent it by $-n$, n being entire or fractional; then $y^m = \frac{1}{y^n}$, and § 80,

$$d(y^m) = -\frac{dy^n}{y^{2n}} = -\frac{ny^{n-1}dy}{y^{2n}} = -n\,y^{-n-1}dy = m\,y^{m-1}dy. \quad \ldots \quad (1).$$

Hence, *the differential of any power of any function with a constant exponent is equal to the product of the exponent of the power, the function with its exponent diminished by unity, and the differential of the* function.

82. Substituting $\frac{1}{n}$ for m in Eq. (1), we have

$$dy^{\frac{1}{n}}=\frac{1}{n}\,y^{\frac{1}{n}-1}\,dy=\frac{1}{n}\,y^{\frac{1-n}{n}}\,dy=\frac{dy}{ny^{\frac{n-1}{n}}}=\frac{dy}{n\sqrt[n]{y^{n-1}}}.$$

Hence, *the differential of the* n^{th} *root of any function is equal to the differential of the function divided by* n *times the* n^{th} *root of the* $n-1$ *power of the function.*

If $n=2$; $\qquad d\sqrt{y}=\dfrac{dy}{2\sqrt{y}}.$

If $n=3$; $\qquad d\sqrt[3]{y}=\dfrac{dy}{3\sqrt[3]{y^2}}.$

Examples.

1. $d\,(2x)^2=2\,(2x)\,d\,(2x)=8xdx.$
2. $d\,(2x^2)^2=2\,(2x^2)\,d\,(2x^2)=16x^3dx.$
3. $d\,4x^4=4\times 4x^3dx=16x^3dx.$
4. $d\,x^n=nx^{n-1}dx.$
5. $d\,(ax)^3=3\,(ax)^2d\,(ax)=3a^3x^2dx.$
6. $d\,(3x)^{-2}=(-2)\,(3x)^{-3}d\,(3x)=-6\,(3x)^{-3}dx.$
7. $d\,x^{-n}=-n\,x^{-n-1}dx.$
8. $d\,x^{\frac{1}{2}}=\frac{1}{2}\,x^{-\frac{1}{2}}\,dx=\dfrac{dx}{2\sqrt{x}}.$
9. $d\,x^{\frac{1}{n}}=\frac{1}{n}\,x^{\frac{1}{n}-1}\,dx=\dfrac{dx}{n\sqrt[n]{x^{n-1}}}.$
10. $d\,x^{-\frac{1}{2}}=-\frac{1}{2}\,x^{-\frac{3}{2}}\,dx=\dfrac{-dx}{2\sqrt{x^3}}.$
11. $d\,x^{\frac{-1}{n}}=-\frac{1}{n}\,x^{-\frac{1}{n}-1}\,dx=\dfrac{-dx}{n\sqrt[n]{x^{n+1}}}.$

83. Differential of log y. Let y be any function of any variable. Increase the variable by h, and denote, by k the corresponding increment of y.

Applying the general rule, § 53, we have, since h and k vanish together,

$$\lim_{k \to 0}\left[\frac{\log(y+k)-\log y}{k}\right] = \lim_{k \to 0}\left[\frac{\log\left[\frac{y+k}{y}\right]}{k}\right] = \lim_{k \to 0}\left[\frac{\log\left(1+\frac{k}{y}\right)}{k}\right],$$

which, placing $k=ym$, equals $\lim_{m \to 0}\left[\frac{\log(1+m)}{ym}\right] =$

$$= \frac{1}{y}\lim_{m \to 0}\left[\log(1+m)^{\frac{1}{m}}\right] = \frac{1}{y}\log e = \frac{M}{y}. \qquad \text{§ 32.}$$

Hence, $\frac{d \log y}{dy} = \frac{M}{y}$; and $\mathbf{d \log y = M \frac{dy}{y}}$.

That is, *the differential of the logarithm of any function is equal to the modulus of the system into the differential of the function divided by the function.*

In the Napierian system, $M=1$, and $d \log_e y = \frac{dy}{y}$.

84. Differential of a^x. Let a be any constant, and x any variable. Increasing x by h, and applying the general rule, § 53, we have

$$\lim_{h \to 0}\left[\frac{a^{x+h}-a^x}{h}\right] = \lim_{h \to 0}\left[a^x \frac{a^h-1}{h}\right] =$$

$$= a^x \lim_{h \to 0}\left[\frac{a^h-1}{h}\right] = a^x \log_e a. \qquad \text{§ 33.}$$

Hence, $\frac{d\,a^x}{dx} = a^x \log_e a$; and $\mathbf{da^x = a^x \log_e a\, dx}$.

That is, *the differential of any exponential function with a constant base is equal to the product of the function, the Napierian logarithm of the base, and the differential of the exponent.*

If $a=e$, the base of the Napierian system, we have $de^x = e^x dx$.

85. Differential of y^z, in which y and z are functions of the same variable.

Let $u=y^z$; then $\log_e u = z \log_e y$; and, § 78,

$$\frac{du}{u} = z\frac{dy}{y} + \log_e y\, dz.$$

Hence, $du = dy^z = z\,y^{z-1}dy + y^z \log_e y\,dz$, which is the sum of the differentials obtained by applying; first, the rule in § 81; then, that in § 84.

86. Logarithmic Differentiation. The differentiation of an exponential function, or one involving a product or quotient, is frequently simplified by first taking the Napierian logarithm of the function as above.

EXAMPLES.

1. $u = x^x$. $\qquad du = x^x(1 + \log_e x)\,dx$.

2. $u = x^{x^x}$. $\qquad du = x^{x^x} \quad x^x \left[\log_e x(\log_e x + 1) + \frac{1}{x}\right] dx$.

3. $u = \frac{\sqrt{1+x}}{\sqrt{1-x}}$. $\qquad du = \frac{dx}{(1-x)\sqrt{1-x^2}}$.

4. $u = x^{\frac{1}{x}}$. $\qquad du = x^{\frac{1-2x}{x}}(1 - \log_e x)\,dx$.

5. $u = \frac{\sqrt{(x-1)^5}}{\sqrt[4]{(x-2)^3}\;\sqrt[3]{(x-3)^7}}$ $\qquad \log_e u = \frac{5}{2}\log_e(x-1) - \frac{3}{4}\log_e(x-2) - \frac{7}{3}\log_e(x-3)$,

$$\frac{du}{u} = \frac{5}{2}\frac{dx}{x-1} - \frac{3}{4}\frac{dx}{x-2} - \frac{7}{3}\frac{dx}{x-3} = -\frac{7x^2+30x-97}{12(x-1)(x-2)(x-3)}\,dx,$$

$$du = -\frac{(x-1)^{\frac{3}{2}}(7x^2+30x-97)}{12(x-2)^{\frac{7}{4}}(x-3)^{\frac{10}{3}}}\,dx.$$

87. $\mathbf{d \sin \varphi = \cos \varphi\, d\varphi}$. For, if φ be increased by $\Delta\varphi$, we shall have, § 53, § 34,

$$\frac{d \sin\varphi}{d\varphi} = \underset{\Delta\varphi \to 0}{\text{limit}}\left[\frac{\sin(\varphi + \Delta\varphi) - \sin\varphi}{\Delta\varphi}\right] = \underset{\Delta\varphi \to 0}{\text{limit}}\left[\frac{2\sin\frac{\Delta\varphi}{2}\cos\left(\varphi + \frac{\Delta\varphi}{2}\right)}{\Delta\varphi}\right] =$$

$$= \underset{\Delta\varphi \to 0}{\text{limit}}\left[\frac{\sin\frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}}\cos\left(\varphi + \frac{\Delta\varphi}{2}\right)\right] = \cos\varphi.$$

In a similar manner, by applying the general rule, § 53, the differential of any trigonometric function may be determined; but it is perhaps simpler to make use of the relations existing between the functions.

$\mathbf{d \cos \varphi = -\sin \varphi\, d\varphi}$. $\qquad$ For, $d\cos\varphi = d\sin\left(\frac{\pi}{2} - \varphi\right) =$

$$= \cos\left(\frac{\pi}{2} - \varphi\right) d\left(\frac{\pi}{2} - \varphi\right) = -\sin\varphi\, d\varphi.$$

$\mathrm{d} \tan \varphi = \frac{\mathrm{d}\, \varphi}{\cos^2 \varphi} = \sec^2 \varphi\, d\, \varphi = (1 + \tan^2 \varphi)\, \mathrm{d}\, \varphi.$ For, $d \tan \varphi = d \frac{\sin \varphi}{\cos \varphi} =$
$= \frac{\cos \varphi\, d \sin \varphi - \sin \varphi\, d \cos \varphi}{\cos^2 \varphi} = \frac{(\cos^2 \varphi + \sin^2)\, d\, \varphi}{\cos^2 \varphi} = \frac{d\, \varphi}{\cos^2 \varphi}.$

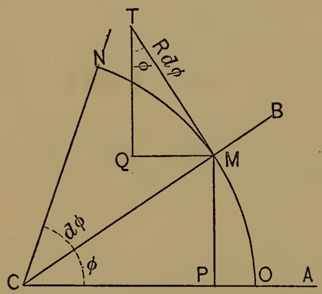
$\mathrm{d} \cot \varphi = -\frac{\mathrm{d}\, \varphi}{\sin^2 \varphi} = -\operatorname{cosec}^2 \varphi\, \mathrm{d}\varphi = -(1 + \cot^2 \varphi)\, \mathrm{d}\varphi.$ For, $d \cot \varphi =$
$= d \tan \left(\frac{\pi}{2} - \varphi\right) = \frac{d\left(\frac{\pi}{2} - \varphi\right)}{\cos^2 \left(\frac{\pi}{2} - \varphi\right)} = -\frac{d\varphi}{\sin^2 \varphi}.$

$\mathrm{d} \sec \varphi = \tan \varphi \sec \varphi d\varphi.$ For, $d \sec \varphi = d \frac{1}{\cos \varphi} = \frac{\sin \varphi\, d\varphi}{\cos^2 \varphi} =$
$= \tan \varphi \sec \varphi\, d\varphi.$

$\mathrm{d} \operatorname{cosec} \varphi = -\cot \varphi \operatorname{cosec} \varphi\, \mathrm{d}\varphi.$ For, $d \operatorname{cosec} \varphi = d \sec \left(\frac{\pi}{2} - \varphi\right) =$
$= \tan \left(\frac{\pi}{2} - \varphi\right) \sec \left(\frac{\pi}{2} - \varphi\right) d \left(\frac{\pi}{2} - \varphi\right) = -\cot \varphi \operatorname{cosec} \varphi d\varphi.$

$\mathrm{d} \operatorname{versin} \varphi = \sin \varphi \mathrm{d}\varphi.$ For, $d \operatorname{versin} \varphi = d(1 - \cos \varphi) = \sin \varphi d\varphi.$

$\mathrm{d} \operatorname{coversin} \varphi = -\cos \varphi \mathrm{d}\varphi.$ For, $d \operatorname{covers} \varphi = d \operatorname{vers} \left(\frac{\pi}{2} - \varphi\right) =$
$= \sin \left(\frac{\pi}{2} - \varphi\right) d \left(\frac{\pi}{2} - \varphi\right) = -\cos \varphi d\, \varphi.$



In order to illustrate the formulas for the differentials of the sin and cos of any angle, let $ACB = \varphi$, be any given angle. Assume $BCN = d\varphi$, and with any radius, as $CO = R$, describe an arc, as OMN.

Then,

$\frac{PM}{R} = \sin \varphi, \quad \frac{CP}{R} = \cos \varphi, \quad \text{arc } MN = Rd\varphi.$

The definition of a differential, § 63, in this case, requires that the sin φ and cos φ, retaining their rates at the states corresponding to $\varphi = ACB$, shall continue to change from those states while φ increases by the angle $BCN = d\varphi$.

Draw the tangent line to the arc at M; and lay off MT equal to the arc $MN = Rd\,\varphi$. Through T draw TQ parallel to MP, and through M, draw MQ parallel to OC.

Then, QT and $-MQ$ are, respectively, the changes that the lines PM and CP would undergo were they to continue to change, with the rates they have when $\varphi = ACB$, while φ increases by $d\varphi$.

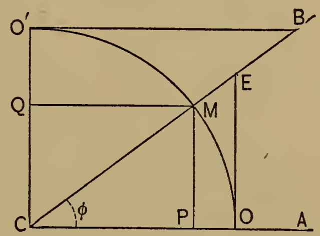
Hence, $\frac{QT}{R}$, and $-\frac{MQ}{R}$ are, respectively, the changes that the sin and cos of φ would undergo under the same requirements.

The angle $MTQ = \varphi$. Hence,

$$QT = MT \cos\varphi = R \cos\varphi\, d\varphi, \quad \text{and} \quad \frac{QT}{R} = d \sin\varphi = \cos\varphi\, d\varphi.$$

$$-MQ = MT \sin\varphi = R \sin\varphi\, d\varphi, \quad \text{and} \quad \frac{MQ}{R} = d \cos\varphi = -\sin\varphi\, d\varphi.$$

In a similar manner let the student illustrate the formulas for the differentials of the other trigonometric functions.



88. Regarding the right lines PM, CP, OE, $O'B$, etc., as functions of the variable angle φ, we have

$$d\,PM = d\,R \sin\varphi = R \cos\varphi d\varphi. \qquad d\,CP = d\,R \cos\varphi = -R \sin\varphi d\varphi$$

$$d\,OE = d\,R \tan\varphi = \frac{R d\varphi}{\cos^2\varphi}. \qquad dO'B = dR \cot\varphi = -\frac{R d\varphi}{\sin^2\varphi}.$$

$$d\,CE = d\,R \sec\varphi = R \tan\varphi \sec\varphi d\varphi. \qquad d\,CB = Rd \operatorname{cosec}\varphi = -R \cot\varphi \operatorname{cosec}\varphi d\varphi.$$

$$d\,OP = d\,R \operatorname{vers}\varphi = R \sin\varphi d\varphi. \qquad d\,O'Q = d\,R \operatorname{covers}\varphi = -R \cos\varphi d\varphi.$$

It is important to notice the difference between the differentials of the above lines, which depend upon the radius of the circle used, and the differentials of the trigonometric functions which do not depend upon any radius or circle.

Examples.

1. $y = \log_e(\sin\varphi)$. $\qquad dy = \frac{d \sin\varphi}{\sin\varphi} = \frac{\cos\varphi\, d\varphi}{\sin\varphi} = \cot\varphi\, d\varphi.$

2. $y = \log_e \sqrt{a^2 - x^2}$. $\qquad dy = -\frac{x dx}{a^2 - x^2}.$

3. $y = e^{nx}$. $\quad e =$ base Napierian system, $\quad dy = ne^{nx} dx.$

4. $y=\log_e \tan\frac{\varphi}{2}$. $\quad dy=\dfrac{d\tan\frac{\varphi}{2}}{\tan\frac{\varphi}{2}}=\dfrac{\frac{d\varphi}{2\cos^2\frac{\varphi}{2}}}{\tan\frac{\varphi}{2}}=\dfrac{d\varphi}{\sin\varphi}$.

5. Assuming $d\varphi=\frac{\pi}{4}$, we have, corresponding to $\varphi=\frac{\pi}{6}$,

$d\sin\varphi=\frac{\sqrt{3}}{8}\pi$, $\quad d\cos\varphi=-\frac{\pi}{8}$, $\quad d\tan\varphi=\frac{\pi}{3}$.

$d\cot\varphi=-\pi$. $\quad d\sec\varphi=\frac{\pi}{6}$. $\quad d\operatorname{cosec}\varphi=-\frac{\sqrt{3}}{2}\pi$.

6. Corresponding to $\varphi=\frac{\pi}{4}$, we have

$\dfrac{d\sin\varphi}{d\varphi}=\dfrac{1}{\sqrt{2}}$. $\quad \dfrac{d\cos\varphi}{d\varphi}=-\dfrac{1}{\sqrt{2}}$. $\quad \dfrac{d\tan\varphi}{d\varphi}=4$.

$\dfrac{d\cot\varphi}{d\varphi}=-4$. $\quad \dfrac{d\sec\varphi}{d\varphi}=\sqrt{2}$. $\quad \dfrac{d\operatorname{cosec}\varphi}{d\varphi}=-\sqrt{2}$.

7. $u=x^{\sin x}$. $\quad du=x^{\sin x}\left[\cos x\log_e x+\dfrac{\sin x}{x}\right]dx$.

8. $u=\dfrac{\sin^m x}{\cos^n x}$. $\quad \therefore\ \log_e u=m\log_e\sin x-n\log_e\cos x$; and

$$\frac{du}{u}=\left[m\frac{\cos x}{\sin x}+n\frac{\sin x}{\cos x}\right]dx \quad \therefore\ du=\left[\frac{m\sin^{m-1}x}{\cos^{n-1}x}+\frac{n\sin^{m+1}x}{\cos^{n+1}x}\right]dx.$$

9. $y=\sin^2\varphi$, $\quad dy=2\sin\varphi\cos\varphi\, d\varphi$.
10. $y=\sin^3\varphi$, $\quad dy=3\sin^2\varphi\cos\varphi\, d\varphi$.
11. $y=\cos^2\varphi$, $\quad dy=-2\cos\varphi\sin\varphi\, d\varphi$.
12. $y=\cos^3\varphi$, $\quad dy=-3\cos^2\varphi\sin\varphi\, d\varphi$.
13. $y=\tan^2\varphi$, $\quad dy=2\tan\varphi\,\dfrac{d\varphi}{\cos^2\varphi}$.
14. $y=\tan^3\varphi$, $\quad dy=3\tan^2\varphi\,\dfrac{d\varphi}{\cos^2\varphi}$.
15. $y=\cot^2\varphi$, $\quad dy=-2\cot\varphi\,\dfrac{d\varphi}{\sin^2\varphi}$.
16. $y=\cot^3\varphi$, $\quad dy=-3\cot^2\varphi\,\dfrac{d\varphi}{\sin^2\varphi}$.
17. $y=\sec^2\varphi$, $\quad dy=2\sec\varphi\tan\varphi\sec\varphi\, d\varphi$.
18. $y=\sec^3\varphi$, $\quad dy=3\sec^2\varphi\tan\varphi\sec\varphi\, d\varphi$.
19. $y=\operatorname{cosec}^2\varphi$, $\quad dy=-2\operatorname{cosec}\varphi\cot\varphi\operatorname{cosec}\varphi\, d\varphi$.
20. $y=\operatorname{cosec}^3\varphi$, $\quad dy=-3\operatorname{cosec}^2\varphi\cot\varphi\operatorname{cosec}\varphi\, d\varphi$
21. $y=\operatorname{versin}^2\varphi$, $\quad dy=2\operatorname{versin}\varphi\sin\varphi\, d\varphi$.
22. $y=\operatorname{versin}^3\varphi$, $\quad dy=3\operatorname{versin}^2\varphi\sin\varphi\, d\varphi$.
23. $y=\operatorname{covers}^2\varphi$, $\quad dy=-2\operatorname{covers}\varphi\cos\varphi\, d\varphi$.
24. $y=\operatorname{covers}^3\varphi$, $\quad dy=-3\operatorname{covers}^2\varphi\cos\varphi\, d\varphi$.

89. 1°. $d\sin^{-1}u = \dfrac{du}{\sqrt{1-u^2}}$. Let $\varphi = \sin^{-1}u$; then $u = \sin\varphi$, and $\dfrac{du}{d\varphi} = \cos\varphi$. Hence, § 73, $\dfrac{d\varphi}{du} = \dfrac{1}{\cos\varphi} = \dfrac{1}{\pm\sqrt{1-\sin^2\varphi}} = \dfrac{1}{\pm\sqrt{1-u^2}}$, or $d\varphi = d\sin^{-1}u = \dfrac{du}{\pm\sqrt{1-u^2}}$*.

2°. $d\cos^{-1}u = \dfrac{-du}{\sqrt{1-u^2}}$. For $d\cos^{-1}u = d\left(\dfrac{\pi}{2} - \sin^{-1}u\right) = \dfrac{-du}{\sqrt{1-u^2}}$,

3°. $d\tan^{-1}u = \dfrac{du}{1+u^2}$. Let $\varphi = \tan^{-1}u$; then $u = \tan\varphi$, and $\dfrac{du}{d\varphi} = \dfrac{1}{\cos^2\varphi}$. Hence, § 73, $\dfrac{d\varphi}{du} = \cos^2\varphi = \dfrac{1}{\sec^2\varphi} = \dfrac{1}{1+\tan^2\varphi} = \dfrac{1}{1+u^2}$, or $d\varphi = d\tan^{-1}u = \dfrac{du}{1+u^2}$.

4°. $d\cot^{-1}u = \dfrac{-du}{1+u^2}$. For, $d\cot^{-1}u = d\left(\dfrac{\pi}{2} - \tan^{-1}u\right) = \dfrac{-du}{1+u^2}$.

5°. $d\sec^{-1}u = \dfrac{du}{u\sqrt{u^2-1}}$. Let $\varphi = \sec^{-1}u$; then $u = \sec\varphi$, and $\dfrac{du}{d\varphi} = \sec\varphi\tan\varphi$. Hence, § 73, $\dfrac{d\varphi}{du} = \dfrac{1}{\sec\varphi\tan\varphi} = \dfrac{1}{\sec\varphi\sqrt{\sec^2\varphi-1}} = \dfrac{1}{u\sqrt{u^2-1}}$, or $d\varphi = d\sec^{-1}u = \dfrac{du}{u\sqrt{u^2-1}}$.

6°. $d\operatorname{cosec}^{-1}u = \dfrac{-du}{u\sqrt{u^2-1}}$. For $d\operatorname{cosec}^{-1}u = d\left(\dfrac{\pi}{2} - \sec^{-1}u\right) = \dfrac{-du}{u\sqrt{u^2-1}}$.

7°. $d\operatorname{versin}^{-1}u = \dfrac{du}{\sqrt{2u-u^2}}$. Let $\varphi = \operatorname{versin}^{-1}u$; then $u = \operatorname{versin}\varphi$, and $\dfrac{du}{d\varphi} = \sin\varphi$. Hence, § 73, $\dfrac{d\varphi}{du} = \dfrac{1}{\sin\varphi} = \dfrac{1}{\sqrt{1-\cos^2\varphi}} = \dfrac{1}{\sqrt{1-(1-\operatorname{vers}\varphi)^2}} = \dfrac{1}{\sqrt{2\operatorname{vers}\varphi - \operatorname{vers}^2\varphi}} = \dfrac{1}{\sqrt{2u-u^2}}$;

or $d\varphi = d\operatorname{vers}^{-1}u = \dfrac{du}{\sqrt{2u-u^2}}$.

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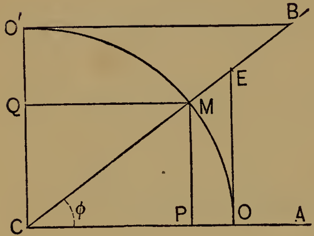
*The sign depends upon that of $\cos\varphi$. The formula is generally written with the plus sign only, which corresponds to angles ending in the 1st or 4th quadrants.

Formulas 2°, 5°, 6°, 7° and 8°, also involve the double sign, but are generally written as indicated above. 2°, 5° and 7°, as given, correspond to angles ending in the 1st or 2nd quadrants; and 6° and 8° correspond to angles ending in the 1st or 4th quadrants.

8°. $d\,\text{covers}^{-1}u = \frac{-du}{\sqrt{2u-u^2}}$. For $d\,\text{covers}^{-1}u = d\left(\frac{\pi}{2} - \text{vers}^{-1}u\right) = \frac{-du}{\sqrt{2u-u^2}}$.



90. Regarding φ as a function of the line PM, denoted by y, we have $\varphi = \sin^{-1}\frac{y}{R}$. Hence,

$$d\varphi = \frac{d\frac{y}{R}}{\sqrt{1-\frac{y^2}{R^2}}} = \frac{dy}{\sqrt{R^2-y^2}}.$$

Similarly, having $CP=y$, $\therefore$ $\varphi = \cos^{-1}\frac{y}{R}$, we have $d\varphi = \frac{-dy}{\sqrt{R^2-y^2}}$.

Similarly, having $OE=y$, $\therefore$ $\varphi = \tan^{-1}\frac{y}{R}$, we have $d\varphi = \frac{Rdy}{R^2+y^2}$.

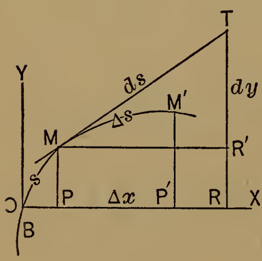
Similarly, having $O'B=y$, $\therefore$ $\varphi = \cot^{-1}\frac{y}{R}$, we have $d\varphi = \frac{-Rdy}{R^2+y^2}$.

Similarly, having $CE=y$, $\therefore$ $\varphi = \sec^{-1}\frac{y}{R}$, we have $d\varphi = \frac{Rdy}{y\sqrt{y^2-R^2}}$.

Similarly, having $CB=y$, $\therefore$ $\varphi = \text{cosec}^{-1}\frac{y}{R}$, we have $d\varphi = \frac{-Rdy}{y\sqrt{y^2-R^2}}$.

Similarly, having $PO=y$ $\therefore$ $\varphi = \text{versin}^{-1}\frac{y}{R}$, we have $d\varphi = \frac{dy}{\sqrt{2Ry-y^2}}$.

Similarly, having $O'Q=y$, $\therefore$ $\varphi = \text{coversin}^{-1}\frac{y}{R}$, we have $d\varphi = \frac{-dy}{\sqrt{2Ry-y^2}}$.



91. Differential of an Arc of a Plane Curve. Let s represent the length of a varying portion of any plane curve in the plane XY. It will be a function of one independent variable only, § 19, which we may take to be x.

Assume any point of the curve, as M, and increase the corresponding value of $x=OP$, by $PP'\,\Delta x$. $\Delta s = MM'$ will be the corresponding increment of s.

Then, § 37,

$$\frac{ds}{dx}=\underset{\triangle x \ggg\!\!\to 0}{\text{limit}}\left[\frac{\triangle s}{\triangle x}\right]=\frac{1}{\cos R'MT}.$$

Assume $PR=dx$. Then, § 63, $R'T=dy$, and $\cos R'MT=\dfrac{dx}{MT}$. Substituting in above,

$$\frac{ds}{dx}=\frac{MT}{dx}, \qquad \therefore \qquad MT=ds=\sqrt{dx^2+dy^2}\,*.$$

The double sign is omitted because s may always be considered as an increasing function of x.

That is, *the differential of an arc of a plane curve is equal to the square root of the sum of the squares of the differentials of the coördinates of its extreme point.*

If s were to change from its state corresponding to any point, as M, with its rate at that state unchanged, the generatrix would move upon the tangent line at M; hence, $MT=\sqrt{dx^2+dy^2}$ represents ds in direction and measure.

In order to express ds in terms of x and dx only, substitute for dy its expression in terms of x and dx determined from the equation of the curve.

Similarly, ds may be expressed in terms of y and dy only. Thus, let s be an arc of the circle whose equation is $x^2+y^2=4$. Solving with respect to y, and differentiating, we have

$$dy=\mp\frac{x\,dx}{\sqrt{4-x^2}}.$$

Hence. $$ds=\sqrt{dx^2+\frac{x^2dx^2}{4-x^2}}=\frac{2\,dx}{\sqrt{4-x^2}}, \quad \text{and} \quad \frac{ds}{dx}=\frac{2}{\sqrt{4-x^2}}.$$

$$\left(\frac{ds}{dx}\right)_{x=-2}=\infty, \qquad \left(\frac{ds}{dx}\right)_{x=0}=1. \qquad \left(\frac{ds}{dx}\right)_{x=2}=\infty.$$

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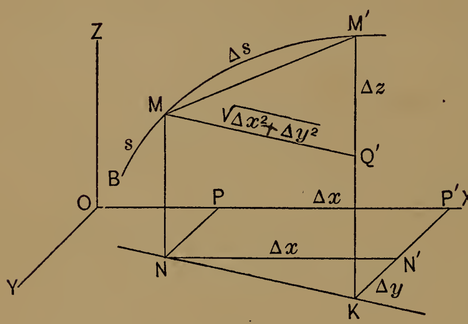
* The square of the differential of a variable represented by a single letter, is generally written as indicated in the above formula; and is similar in form to the symbol for the differential of the square of the variable.

Similarly, the n^{th} power of the dx is generally written dx^n.

A knowledge of the formula, and of the associated symbols, removes the ambiguity in such cases.

That is, at the points where the circum. cuts X the rate of s with respect to x is infinity; while at the points where it intersects Y its rate is the same as that of x.



92. Differential of any Arc. Let s represent the length of a varying portion of any curve in space. It will be a function of one independent variable only, § 19, which we may assume to be x.

Through any assumed point of the curve, as M, draw the ordinate MN; and through N, the point where it pierces XY, draw NP parallel to Y. OP will be the value of x corresponding to M. Increase $x=OP$ by $PP'=\Delta x$, and through P' pass a plane parallel to YZ, intersecting the given curve at M'. $\Delta s=$ arc MM' will be the increment of s corresponding to the assumed increment of x.

Draw the chord MM' and the ordinate $M'K$. Through M draw MQ' parallel to a right line drawn through N and K; and through N draw NN' parallel to X. Then, $N'K=\Delta y$, and $Q'M'=\Delta z$, will be the increments of y and z corresponding to Δx; and we have chord $MM'=\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$.

Hence, § 31, Theorem X.

$$\frac{ds}{dx}=\underset{\Delta x \ggg\!\to 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta x}\right]=\underset{\Delta x \ggg\!\to 0}{\text{limit}}\left[\frac{\text{chord } MM'}{\Delta x}\right]$$

$$=\underset{\Delta x \ggg\!\to 0}{\text{limit}}\left[\frac{\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}}{\Delta x}\right]$$

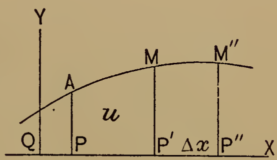
$$=\underset{\Delta x \ggg\!\to 0}{\text{limit}}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2+\left(\frac{\Delta z}{\Delta x}\right)^2}$$

$$=\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}\ ; \text{ hence, } \quad \mathbf{ds=\sqrt{dx^2+dy^2+dz^2}}.$$

In the above deduction s may be a curve of single, or of double curvature. The increment Δs may, or may not, lie in the projecting plane of the chord MM'.

If not, the projection of the chord MM' on the plane XY will change direction as Δx approaches zero, but the above relations will not be affected thereby.

93. Differential of a Plane Area. Let u represent the area of the plane surface included between any varying portion of any plane curve, as AM, the ordinates of its extremities, and the axis of X.

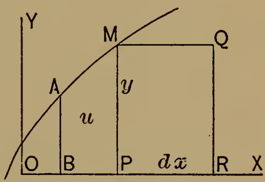


Regarding u as a function of x, § 21; let $x = QP'$, be increased by $P'P'' = \Delta x$. $P'MM''P''$ will be the corresponding increment of u. Hence, § 38,

$$\frac{du}{dx} = \underset{\Delta x \ggg\!\!\rightarrow 0}{\text{limit}} \left[\frac{P'MM''P''}{\Delta x} \right] = y. \quad \therefore \quad \mathbf{du = y\ dx}^*.$$

That is, *the differential of a plane area is equal to the ordinate of the extreme point of the bounding curve into the differential of the abscissa.*

To illustrate, let u represent the area $BAMP$, and $PR = dx$; then $du = ydx =$ rect. PQ, which fulfils the requirements of the definition of a differential, § 63.



Similarly, it may be shown that **xdy** is the differential of the plane area included between any arc, the abscissas of its extremities, and the axis of Y.

In case the coördinate axes are inclined to each other by an angle θ, we have $du = y \sin \theta dx$, or $du = x \sin \theta dy$.

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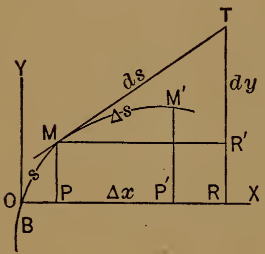
*It is important to notice and remember that ydx is the differential of a plane area bounded as described; and that it is not, in general, the differential of a plane area otherwise bounded.

In order to express du in terms of x and dx, substitute for y, or dy, its expression determined from the equation of the bounding curve.

Thus, if $a^2y^2+b^2x^2=a^2b^2$ is the equation of the bounding curve, we have $y=\frac{b}{a}\sqrt{a^2-x^2}$; and $du=\frac{b}{a}\sqrt{a^2-x^2}\,dx$.



94. Differential of a Surface of Revolution. Let the axis of X coincide with the axis of revolution; and let $BM=s$, be any varying portion of the meridian curve in the plane XY. Through M draw the tangent MT, the ordinate MP, and the right line MR' parallel to X. Let u represent the surface generated by s; and regarding it as a function of x, § 23, let $x=OP$ be increased by $PP'=\Delta x$. $MM'=\Delta s$, will be the corresponding increment of s; and the surface generated by it will be the increment of the function u corresponding to Δx. Hence, § 39,

$$\frac{du}{dx}=\underset{\Delta x \Rightarrow 0}{\text{limit}}\left[\frac{\text{sur. gen. by arc } MM'}{\Delta x}\right]=\frac{2\pi y}{\cos R'MT}.$$

Assume $PR=dx$; then $R'T=dy$, $MT=ds$, and $\cos R'MT=\frac{dx}{ds}$. Substituting this expression for $\cos R'MT$ in above, we have

$$\frac{du}{dx}=\frac{2\pi y\,ds}{dx}; \text{ and } du=2\pi y\,ds=2\pi y\sqrt{dx^2+dy^2}.$$

Hence, *the differential of a surface of revolution is equal to the product of the circum. of a circle perpendicular to the axis and the differential of the arc of the generating curve.*

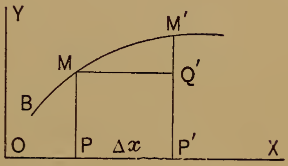
Similarly, it may be shown that $2\pi x\sqrt{dx^2+dy^2}$ is the differential of a surface of revolution generated by revolving a plane curve about the axis of Y.

In order to express du in terms of a single variable and its differential, find expressions for y and dy in terms of x and dx, or of dx in terms of y and dy, from the equation of the generating curve; and substitute them in the formula.

Thus, if $y^2=2px$ is the equation of the generating curve, we have $y=\sqrt{2px}$ and $dy=\frac{p\,dx}{\sqrt{2px}}$. Hence,

$$du=2\pi\sqrt{2px}\sqrt{dx^2+\frac{p^2dx^2}{2px}}=2\pi\,(2px+p^2)^{\frac{1}{2}}dx\,.$$

95. Differential of a Volume of Revolution. Let the axis of X coincide with the axis of revolution; and let BM be any varying portion of the meridian curve in the plane XY. Through M draw the ordinate MP, and the right line MQ' parallel to X.



Let v represent the volume generated by the plane surface included between the arc BM, the ordinates of its extremities, and the axis of X. Regarding v as a function of x, § 27, let x be increased by $PP'=\Delta x$. The volume generated by the plane surface $PMM'P'$ will be the corresponding increment of the function v.

Then, § 40,

$$\frac{dv}{dx}=\underset{\Delta x \ggg\!\!\to 0}{\text{limit}}\left[\frac{\text{vol. gen. by } PMM'P'}{\Delta x}\right]=\pi y^2; \quad \text{and} \quad \mathbf{dv=\pi y^2dx.}$$

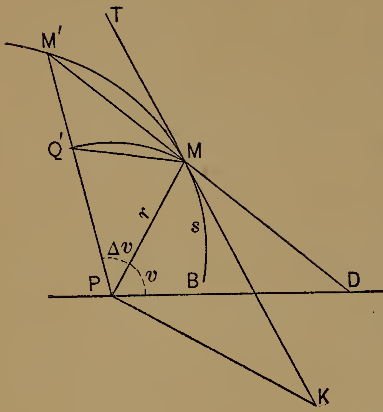
Hence, *the differential of a volume of revolution is equal to the area of a circle perpendicular to the axis into the differential of the abscissa of the meridian curve.*

Similarly, it may be shown that πx^2dy is the differential of a volume of revolution generated by revolving a plane surface about the axis of Y.

In order to express dv in terms of a single variable and its differential, determine an expression for y in terms of x, or of dx in terms of y and dy, from the equation of the meridian curve; and substitute them in the formula.

Thus, if $x^2+y^2-2Rx=0$ is the equation of the meridian curve, we have $dv=\pi\,(2Rx-x^2)\,dx;$ or since

$$dx=\frac{y\,dy}{\sqrt{y^2+R^2}}, \qquad dv=\frac{\pi y^3dy}{\sqrt{y^2+R^2}}.$$

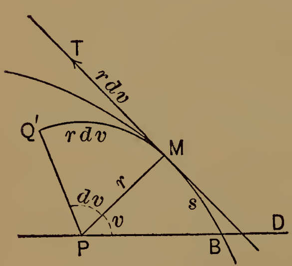


96. Differential of an Arc of a Plane Curve in terms of Polar Coördinates. Let $r=f(v)$ be the polar equation of any plane curve, as BMM', referred to the fixed right line PD, and the pole P. Let $BM=s$, be any varying portion of the curve, and $PM=r$, the radius vector corresponding to M. Regarding s as a function of v, § 19, let v be increased by $MPM'=\Delta v$. The arc $MM'=\Delta s$ will be the corresponding increment of s. With P as a centre, and PM as a radius, describe the arc MQ'. Denote PM' by r'; then $Q'M'=r'-r$, will be the increment of r corresponding to Δv. Through M draw the tangent MT, and the chords MM' and MQ'.

Then, § 41, we have

$$\frac{ds}{dv}=\underset{\Delta v \ggg\rightarrow 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta v}\right]=\underset{\Delta v \ggg\rightarrow 0}{\text{limit}}\sqrt{\left(\frac{r'-r}{\Delta v}\right)^2+r^2}=\sqrt{\frac{dr^2}{dv^2}+r^2},$$

Hence, $$ds=\sqrt{dr^2+r^2dv^2}.$$

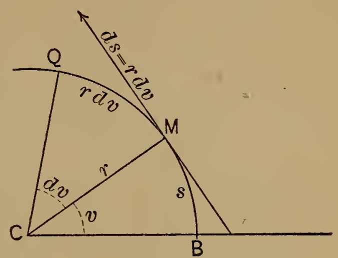


If the radius vector PM coincides with the normal to the curve at M, the corresponding tangent to the arc MQ' will coincide with MT; and § 41,

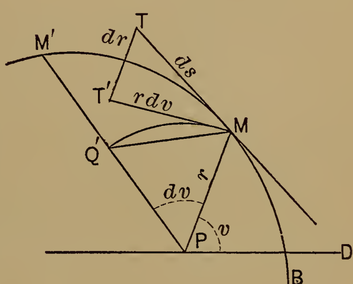
$$\frac{ds}{dv}=\underset{\Delta v \ggg\rightarrow 0}{\text{limit}}\left[\frac{\text{arc } MM'}{\Delta v}\right]=r, \text{ giving } ds=rdv.$$

In this case $dr=0$, because the motion of the generatrix at the point considered is perpendicular to the radius vector.

Since the radius of a circle is always normal to the arc, *the differential of an arc of a circle regarded as a function of the corresponding angle at the centre, is equal to its radius into the differential of the angle.*



Let $BM=s$, be an arc of a circle subtending the angle $BCM=v$. Assume $MCQ=dv$; then will the arc $MQ=rdv$. The direction of the motion of the generatrix at any point, is along the corresponding tangent to s; hence, by laying off from M, upon the tangent at that point, a distance equal to $ds=$arc $MQ=rdv$, we have ds represented in measure and direction.



In order to represent ds in the general case, let BM be the given curve, P the pole, M the assumed point, and $MPM'=dv$. If r were constant, as we have seen in the case of a circle, $MT'=rdv$ would be ds; but, in general, ds is affected by a uniform change in r, in the direction PM, equal to dr. To determine it, we have

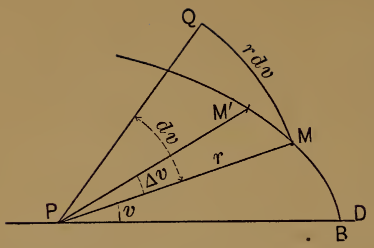
$$\frac{dr}{dv}=\lim_{\triangle v \ggg\!\rightarrow 0}\left[\frac{Q'M'}{\triangle v}\right]=\lim_{\triangle v \ggg\!\rightarrow 0} r\left[\frac{Q'M'}{Q'M}\right]=r\tan T'MT.$$

At T' draw $T'T$ parallel to PM; then $\dfrac{T'T}{T'M}=\tan T'MT=\dfrac{T'T}{rdv}$.

Hence, $\dfrac{dr}{dv}=r\dfrac{T'T}{rdv}=\dfrac{T'T}{dv}$, and $dr=T'T$.

Hence, $MT=ds=\sqrt{dr^2+r^2dv^2}$, represents ds in measure and direction.

In order to express ds in terms of a single variable and its differential, find expressions for r and dr in terms of v and dv, or an expression for dv in terms of r and dr, from the polar equation of the curve; and substitute in the formula.



97. Differential of a Plane Area in terms of Polar Coördinates. Let u represent the area of a varying portion of the surface generated by the radius vector PM revolving about the pole P. Regarding u as a function of v, § 22, let $MPM' = \Delta v$. The area MPM', represented by Δu, will be the corresponding increment of u.

Hence, § 42,

$$\frac{du}{dv} = \underset{\Delta v \ggg\!\!\to 0}{\text{limit}} \left[\frac{\Delta u}{\Delta v}\right] = \frac{r^2}{2}; \quad \text{and} \quad du = \frac{r^2 dv}{2}.$$

To illustrate, with $PM = r$, describe the arc of a circle $MQ = rdv$ corresponding to $MPQ = dv$; then $du = \frac{r^2 dv}{2} =$ area of the circular sector MPQ.

du may be expressed in terms of v and dv, by substituting for r its value in terms of v, determined from the polar equation of the bounding curve.

Problems in Rates.

1. Having $s^2 = 5t^3$, find the velocity and acceleration when $t = 2$ seconds; $t = 3$ seconds.

2. Find the angles that a tangent to the curve $x^2 = 6y^2 + 3y + 1$, at the point (8, 3), makes with the axes X and Y, respectively.

3. Find the rate of change of $\left(\sqrt{x} + \frac{3}{ax^2}\right)$ when $x = 3$.

4. Find the angles that a tangent to the curve $y = \log x$, at the point (1, 0) makes with the coördinate axes respectively.

5. Find the rate of change of the ordinate of a circle with respect to the abscissa.

6. Same of an ellipse.

7. Same of a parabola.

8. Same of an hyperbola.

9. At what rate does the volume of a cube change with respect to the length of an edge?

10. Find the rate of change of a logarithm in the common system when the number is 12.

11. Same for the numbers $\frac{1}{4}$, $\frac{1}{3}$, 157, 3227.

12. The area of a circle is increasing 5 sq. ft. a second; find the rate per second of its radius when the radius is 3 feet.

12. The side of a square is increasing 3 in. a minute; find the rate per minute of its area.

14. The relation between the time denoted by t; and the distance, represented by s, through which a body, starting from rest, falls in a vacuum near the earth's surface, is expressed, very nearly, by the equation $s=16.1\,t^2$; s being in feet and t in seconds. Construct a table giving, the entire distance fallen through in 1 second; in 2 seconds; in 3 seconds; and in 4 seconds; the distance passed over during each of the above seconds: the velocity and acceleration at the end of each.

Time in Seconds.	Entire Distance in Feet.	Distance each Second.	Velocity.	Acceleration.
1	16.1	16.1	32.2	32.2
2	64.4	48.3	64.4	32.2
3	144.9	80.5	96.6	32.2
4	257.6	112.7	128.8	32.2

The following general outline of steps may assist the student in solving problems involving rates.

1°. Draw a figure representing the magnitudes and directions under consideration; and denote the variable parts by the final letters of the alphabet.

2°. Following the word *given*, write, with the proper symbols, all known data; and after the word *required* indicate the symbols for the required rates.

3°. From the relations between the magnitudes, find an expression for the function whose rate is required, in terms of the variable.

4°. Differentiate and determine values or expressions for the required rates.

In case an explicit function of a variable cannot be found, make use of the principles in § 74.

15*. A man 6 feet in height, walks away from a light 10 feet above the ground, at the rate of 3 mi. per hour. At what rate is the end of his shadow moving, and at what rate does his shadow increase in length?

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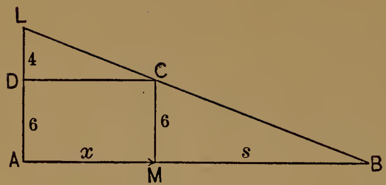
* Examples 15 to 22 are from Rice and Johnson's Calculus.



Let $x = AM$ = distance from foot of light to man.

Let $y = AB$ = distance from foot of light to end of shadow.

Let $s = MB$ = length of shadow.

Let t = time in hrs.

Given, $AL = 10$ ft., $MC = 6$ ft., $DL = 4$ ft., $\dfrac{dx}{dt} = 3\dfrac{\text{mi.}}{\text{hr.}}$ *.

Required, $\dfrac{dy}{dt}$, and $\dfrac{ds}{dt}$.

From similar triangles,

$$y : x :: 10 : 4, \qquad \therefore \qquad y = \frac{5}{2}x, \quad \text{and} \quad \frac{dy}{dx} = \frac{5}{2}.$$

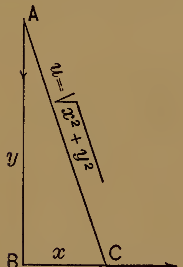
§ 74, $$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{5}{2} \times \frac{3\text{ mi.}}{\text{hr.}} = 7\tfrac{1}{2}\,\frac{\text{mi.}}{\text{hr.}}.$$

Also,

$$s : x :: 6 : 4, \qquad \therefore \qquad s = \frac{3}{2}x, \quad \text{and} \quad \frac{ds}{dx} = \frac{3}{2}.$$

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt} = \frac{3}{2} \times 3\,\frac{\text{mi.}}{\text{hr.}} = 4\tfrac{1}{2}\,\frac{\text{mi.}}{\text{hr.}}.$$

16. A vessel sailing south at the rate of 8 mi. per hour, is 20 mi. north of a vessel sailing east at the rate of 10 mi. an hour. At what rate are they separating at the time? at the end of $1\frac{1}{2}$ hrs.? at the end of $2\frac{1}{2}$ hrs.? When are they neither separating from nor approaching each other?



Let t = time in hours from the given epoch.

Let $AB = y = 20 - 8t$ = distance of 1st ship from BC t hours after the given epoch.

Let $BC = x = 10t$ = distance of 2nd ship from BA at the same time.

Let $u = AC = \sqrt{x^2 + y^2} = \sqrt{400 - 320t + 164t^2}$.

Given, $\dfrac{dy}{dt} = -8\,\dfrac{\text{mi.}}{\text{hr.}}$; $\dfrac{dx}{dt} = 10\,\dfrac{\text{mi.}}{\text{hr.}}$.

Required, $$\frac{du}{dt} = \frac{-160 + 164t}{\sqrt{400 - 320t + 164t^2}},$$

$$\left(\frac{du}{dt}\right)_{t=0} = -8\,\frac{\text{mi.}}{\text{hr.}}. \qquad \left(\frac{du}{dt}\right)_{t=1\frac{1}{2}} = 5\frac{1}{17}\,\frac{\text{mi.}}{\text{hr.}}. \qquad \left(\frac{du}{dt}\right)_{t=\frac{40}{41}} = 0.$$

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* $3\,\dfrac{\text{mi.}}{\text{hr.}}$ indicates 3 mi. per hour.

17. The rate of increase of a side of an equilateral triangle is ½ inch per second, find the rate of its altitude per second. If the rate of a side is 3 feet per second, find the rate per second of the area when the side is 10 ft.

18. A man walks on a straight line, 5 ft. per second. How fast does he approach a point 120 ft. from his path in a perpendicular to it, when he is 50 ft. from the foot of the perpendicular? $1\frac{12}{13}\ \frac{\text{ft.}}{\text{sec.}}$

19. A ladder 25 ft. in length leans against a wall; the bottom is drawn out 2 feet per second, at what rate is the top descending when the bottom is 7 ft. from the wall? $7\ \frac{\text{in.}}{\text{sec.}}$

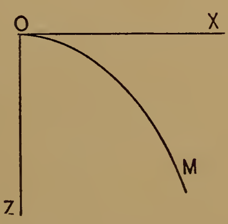
20. Two locomotives are moving along two straight railways which intersect at an angle of 60°; one approaches the intersection at 25 miles per hour, and the other is leaving it at the rate of 30 miles an hour, find rate per hour at which they are separating from each other when each is 10 miles from the intersection. $2\frac{1}{2}\ \frac{\text{mi.}}{\text{hr.}}$

21. A street crossing is 10 ft. from a lamp situated directly over the curbstone, which is 60 ft. from walls of opposite buildings. If a man walks across to opposite side at the rate of 4 miles per hour, at what rate per hour will his shadow move upon the walls when he is 5 ft. from the curbstone? When he is 20 ft. from the curbstone? $96\ \frac{\text{mi.}}{\text{hr.}}\quad 6\ \frac{\text{mi.}}{\text{hr.}}$.

22. The radius of a sphere is decreasing 2 in. per second; find the rate of its surface, and volume.

23. In the parabola $y^2=9x$, find the rate of y with respect to x when $x=4$. What value will x have when rate of y equals that of x? When rate of y is the greatest? When the least?

24*. A boy is running on a horizontal plane towards the foot of a tower 60 ft. in height. How much faster is he approaching the foot than the top of the tower? How far is he from the foot when he is approaching it twice as fast as he is the top? At 100 feet from foot, how much faster is he approaching it than the top?



25. $x^2=2pz$ is the equation of a parabola OM. A point starting from O, moves along the curve in such a manner that $z=16.1t^2$; in which z is expressed in ft., and t in seconds. Find the rate of x with respect to t.

Given,

$$\frac{dx}{dz}=\frac{p}{\sqrt{2pz}}=\frac{p}{\sqrt{32.2pt}}. \qquad \frac{dz}{dt}=32.2\,t,$$

* From Olney's Calculus.

Required, $\frac{dx}{dt}$.

$$\frac{dx}{dt}=\frac{dx}{dz}\times\frac{dz}{dt}=\frac{p}{\sqrt{32.2\,pt^2}}\times 32.2\,t=\sqrt{32.2\,p}.$$

26. One ship was sailing south at $6\,\frac{\text{mi.}}{\text{hr.}}$, another east at $8\,\frac{\text{mi.}}{\text{hr.}}$. At 4 P.M. the second crossed the tracks of the first at a point where the first was 2 hrs. before. How was the distance between the ships changing at 3 P. M.? When was the distance between them not changing?

27. A ship is sailing south 60° east, $8\,\frac{\text{mi.}}{\text{hr.}}$; find the rate of her latitude and longitude.

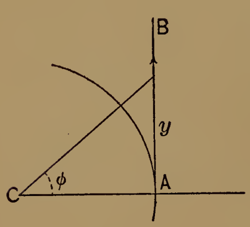
28. A point P moves in a straight line away from a point B at the rate of $8\,\frac{\text{mi.}}{\text{hr.}}$; find its velocity with respect to a point C situated upon the perpendicular to the line BP through B and at 100 ft. from B, when $BP=50$ ft. when $BP=150$ ft.

29. If a circular plate of metal expands by heat so that its diameter increases uniformly at the rate of $\frac{1}{100}$ of an inch per second, at what rate is its surface increasing when the diameter is 2 inches? $\frac{\pi}{100}\,\frac{\text{sq. in.}}{\text{sec.}}$.

30. If the diameter of a sphere increases uniformly at the rate of $\frac{1}{10}$ inches per second, what is its diameter when the volume is increasing at the rate of 5 cubic inches per second? $\frac{10}{\sqrt{\pi}}$ in.

31. If the diameter D of the base of a cone increases uniformly at the rate of $\frac{1}{10}$ inch per second, at what rate is its volume increasing when D becomes 10 inches, the height being constantly one foot? $2\,\frac{\text{cu. in.}}{\text{sec.}}$.

32*. Water is poured into a conical glass, 3 inches in height, at a uniform rate, filling the glass in 8 seconds. At what rate is the surface rising at the end of 1 second? At what rate when the surface reaches the brim? $\frac{1}{2}\,\frac{\text{in.}}{\text{sec.}}$ $\frac{1}{8}\,\frac{\text{in.}}{\text{sec.}}$.



33. A train is running from A to B at the rate of 20 mi. an hour. The distance from A to C, on a perpendicular to $A\,B$, is two mi. Find the rate of the angle at C included between $C\,A$ and a right line from C to the train.

Let $\varphi=$ variable angle at C.

" $y=$ distance from A to train.

Given. $CA=2$ mi., $\frac{dy}{dt}=20\,\frac{\text{mi.}}{\text{hr.}}$.

* Rice and Johnson's Calculus.

Required. $\frac{d\varphi}{dt}$.

$y = CA \tan\varphi \quad \therefore \quad \varphi = \tan^{-1}\frac{y}{CA}$ and $\frac{d\varphi}{dy} = \frac{CA}{\overline{AC}^2 + y^2} = \frac{2}{4+y^2}$.

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dy} \times \frac{dy}{dt} = \frac{2}{4+y^2} \times 20 = \frac{40}{4+y^2}.$$

$$\left(\frac{d\varphi}{dt}\right)_{y=0} = 10. \qquad \left(\frac{d\varphi}{dt}\right)_{y=2} = 5. \qquad \left(\frac{d\varphi}{dt}\right)_{y=\infty} = 0.$$

The unit of measure of φ is a radian.

34. In a right plane triangle one side adjacent to the right angle is constant and 4 mi. in length; the other side adjacent to the right angle, denoted by y, is variable. Let φ represent the angle opposite y. Find the rate of φ; first, when $\varphi = f(y)$; also, when $\varphi = F(\tan\varphi)$, corresponding to $y = 2$ mi., and explain the difference between the two results.

Let $u = \tan\varphi = \frac{y}{R} = \frac{y}{4}$.

1°. $d\varphi = \frac{Rdy}{R^2+y^2} \quad \therefore \quad \left(\frac{d\varphi}{dy}\right)_{y=2} = \frac{1}{5}$. $\qquad 2^\circ$. $d\varphi = \frac{du}{1+u^2} \quad \therefore \quad \left(\frac{d\varphi}{du}\right)_{y=2} = \frac{4}{5}$.

φ changes less rapidly with respect to y than it does with respect to u, because y changes 4 times as fast as u.

35. Determine the manner in which the sin of an angle varies with the angle.

$$\frac{d \sin\varphi}{d\varphi} = \cos\varphi = \text{rate} = r, \qquad \frac{dr}{d\varphi} = -\sin\varphi.$$

As φ increases from O to $\frac{\pi}{2}$, the rate is +, but diminishing. Hence, the sin increases, but its increments decrease.

From $\frac{\pi}{2}$ to π, the rate is —, and diminishing. Hence, the sin diminishes and its decrements increases numerically.

From π to $\frac{3\pi}{2}$, the rate is—, and increasing. That is, the sin decreases, but its decrements diminish numerically.

From $\frac{3\pi}{2}$ to 2π, the rate is +, and increasing. That is, the sin increases, and its increments increase.

In a similar manner determine the circumstances of change of each trigonometric function, with respect to the angle.

36. Determine the rate of change of the tangent, regarded as a function of the sine of an angle.

Same of the sin as a function of the cos.
Same of the sin as a function of the sec.
Same of the cos as a function of the cot.
Same of the cos as a function of the cosec.
Same of the tan as a function of the sec.
Same of the tan as a function of the versin.
Same of the cosec. as a function of the covers.
Same of the sec as a function of the vers.

37*. Two points start together from an extremity of a diameter of a circle whose radius is 150 feet. One point moves uniformly along the diameter at the rate of 5 ft. per second; the other moves in the circum. and is always in the perpendicular to the diameter through the first point. Find the velocity of the second point when the angle subtended by the arc described by it is 45°.

38*. Two points start as in above example; one moving uniformly along the tangent at the rate of 10 ft. per second and the other in the circum. so as to be always in the right line joining the first with the centre of the circle. Find the velocity of the second when passing the 45° point.

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* From Bowser's Calculus.

CHAPTER VI.

DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE VARIABLES.

98. The Partial Differential of a Function of Two or more Variables, with respect to one of the variables, is the change that the function would undergo from any state, were it to retain its rate at that state, with respect to that variable, while that variable changed by its differential.

The Total Differential of a Function of Two Variables is the change that the function would undergo from any state, were it to retain its rate at that state, with respect to each variable, while both variables changed by their differentials.

Any function of two variables which changes uniformly with each variable, has a constant rate with respect to each, and its form must be some particular case of the general expression $Ax+By+C$, § 59.

Representing such a function by z, we have

$$z=Ax+By+C \quad . \quad . \quad . \quad . \quad (1).$$

Increasing x and y by their differentials, and denoting the corresponding new state of the function by z', we have

$$z'=A(x+dx)+B(y+dy)+C \quad . \quad . \quad . \quad . \quad (2).$$

Subtracting (1) from (2), member from member, we have

$$z'-z=A\,dx+Bdy.$$

Since the function z changes uniformly with respect to each variable, the total differential of it, denoted by dz, is equal to the corresponding change in the function.

Therefore, $$dz=Adx+Bdy.$$

Adx is the corresponding partial differential of the function z with respect to x; and Bdy is the same with respect to y.

Hence, *the total differential of any function of two variables, which changes uniformly with respect to each, is equal to the sum of the corresponding partial differentials.*

The total differential of any function of two variables which does not vary uniformly with each variable, is not, in general, the corresponding change in the function, but it is the corresponding change of a function having a constant rate with respect to each variable, equal to that of the given function at the state considered. In other words, the total differential is equal to *that* of a function *which changes uniformly* with each variable, and which has at the state considered its partial differentials equal to the corresponding partial differentials of the given function.

Hence, *the total differential of any function of two variables is equal to the sum of the corresponding partial differentials.*

That is, having $z=f(x,y)$, then $dz=\frac{dz}{dx}dx+\frac{dz}{dy}dy.$

In a similar manner it may be shown that, *the total differential of any function of any number of variables is equal to the sum of the corresponding partial differentials.*

Examples.

1. $d(xy)=xdy+ydx.$
2. $d(3ax^2y-2y^2+3bx^3-5)=6axydx+9bx^2dx+3ax^2dy-4ydy.$
3. $d\left(\frac{x+y}{x-y}\right)=\frac{2(xdy-ydx)}{(x-y)^2}.$
4. $d(x^2y^2z^2)=2y^2z^2xdx+2x^2z^2ydy+2x^2y^2zdz.$
5. $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}.$
6. $d[\sin(xy)]=\cos(xy)(ydx+xdy).$
7. $d\log_e(x^y)=\frac{ydx}{x}+\log_e xdy.$
8. $dy^{\sin x}=y^{\sin x}\left(\log_e y\cos xdx+\frac{\sin x\, dy}{y}\right).$
9. $d\operatorname{versin}^{-1}\frac{x}{y}=\frac{ydx-xdy}{y\sqrt{2xy-x^2}}.$
10. $d\sin(x+y)=\cos(x+y)(dx+dy).$

11. $z=\tan^{-1}\frac{x}{y}$, $y^2+x^2=a^2$. Required $\frac{dz}{dx}=\frac{1}{\sqrt{a^2-x^2}}$.

12. $u=y^2+x^2-a^2=0$. Required $du=0$, and $\frac{dy}{dx}=-\frac{x}{y}$.

13. $u=yz$, $y=e^x$, $z=x^4-4x^3+12x^2-24x+24$. Required $\frac{du}{dx}=e^x x^4$.

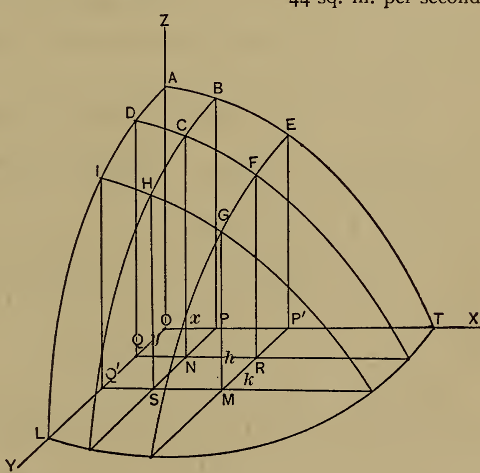
14. $u=\sin^{-1}(p-q)$, $p=3x$, $q=4x^3$. Required $\frac{du}{dx}=\frac{3}{\sqrt{1-x^2}}$.

15. Deduce the formula $ds=\sqrt{dr^2+r^2dv^2}$, § 95, from the formulas,

$$\left.\begin{array}{l} x=a+r\cos v \\ y=b+r\sin v \end{array}\right\}$$ [Anal. Geo.]; and $ds=\sqrt{dx^2+dy^2}$, § 90.

16. One side of a rectangle increases at the rate of 3 in. per second and the other decreases at the rate of 2 in. per second. Find the rate of the area, when the first side is 10 in. and the second 8 in. in length.

44 sq. in. per second.



99. Let ATL be any surface, and $ABCD=u$ a portion of it included between the coördinate planes XZ, YZ, and the planes DQR BPS, parallel to them respectively. From § 25, we have $u=f(x,y)$.

Increase $OP=x$ by $PP'=h$ giving, § 53,

$$\frac{du}{dx}=\underset{h\ggg\to 0}{\text{limit}}\left[\frac{f(x+h,y)-f(x,y)}{h}\right].$$

Now increase $OQ=y$ by $QQ'=k$, giving, § 53,

$$\frac{d^2u^*}{dx\,dy}=\underset{k\ggg\to 0}{\text{limit}}\left[\frac{\underset{h\ggg\to 0}{\text{limit}}\left[\frac{f(x+h,y+k)-f(x,y+k)-[f(x+h,y)-f(x,y)]}{h}\right]}{k}\right], \text{ or}$$

* $\frac{d^2u}{dx\,dy}$ is a symbol for the partial differential coefficient of $\frac{du}{dx}$, taken with respect to y.

$$\frac{d^2u}{dx\,dy}=\underset{\substack{h\ggg\to 0\\ k\ggg\to 0}}{\text{limit}}\left[\frac{f(x+h,y+k)-f(x,y+k)-[f(x+h,y)-f(x,y)]}{h\cdot k}\right].$$

In which,

$f(x+h,y+k)=AEGI.$ $f(x,y+k)=ABHI.$

$f(x+h,y)=AEFD.$ $f(x,y)=ABCD.$

Hence,

$$f(x+h,y+k)-f(x,y+k)=AEGI-ABHI=BEGH.$$

$$f(x+h,y)-f(x,y)=AEFD-ABCD=BEFC.$$

$$f(x+h,y+k)-f(x,y+k)-[f(x+h,y)-f(x,y)]=BEGH-BEFC=CFGH.$$

Therefore, $\dfrac{d^2u}{dx\,dy}=\underset{\substack{h\ggg\to 0\\ k\ggg\to 0}}{\text{limit}}\left[\dfrac{CFGH}{h\,k}\right].$

Let $CF'G'H'$ (not drawn) be the portion of the tangent plane to the surface at C included between the same planes that determine $CFGH$, and let β represent the angle between the tangent plane and XY.

Then $CF'G'H'\times\cos\beta=NRMS=hk$, § 43, or $CF'G'H'=\dfrac{h\,k}{\cos\beta}$.

From § 44, we have $\underset{\substack{h\ggg\to 0\\ k\ggg\to 0}}{\text{limit}}\left[\dfrac{CF'G'H'}{CFGH}\right]=1.$

Hence § 31, Theorem X,

$$\frac{d^2u}{dx\,dy}=\underset{\substack{h\ggg\to 0\\ k\ggg\to 0}}{\text{limit}}\left[\frac{CF'G'H'}{h\,k}\right]=\underset{\substack{h\ggg\to 0\\ k\ggg\to 0}}{\text{limit}}\left[\frac{\frac{h\,k}{\cos\beta}}{h\,k}\right]=\frac{1}{\cos\beta},$$

and

$$\frac{d^2u}{dx\,dy}dx\,dy=\frac{dx\,dy}{\cos\beta}.$$

The same formula may be deduced in a similar manner from the figure without using the functional notation.

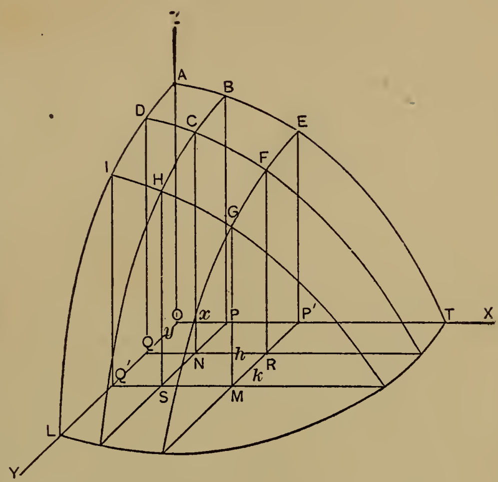
Thus, $\dfrac{du}{dx}=\underset{h\ggg\to 0}{\text{limit}}\left[\dfrac{BCFE}{h}\right].$ Increasing y by $QQ'=k$, we have

$$\frac{d^2u}{dx\,dy}=\underset{k\ggg\to 0}{\text{limit}}\left[\frac{\text{Increment due to } k \text{ of}\left(\underset{h\ggg\to 0}{\text{limit}}\left[\frac{BCFE}{h}\right]\right)}{k}\right]$$

$$=\underset{k\ggg\to 0}{\text{limit}}\left[\frac{\underset{h\ggg\to 0}{\text{limit}}\left[\text{Increment due to } k \text{ of } \frac{BCFE}{h}\right]}{k}\right]=\underset{\substack{k\ggg\to 0\\ h\ggg\to 0}}{\text{limit}}\left[\frac{CFGH}{h\,k}\right]$$

$$=\underset{\substack{k\ggg\to 0\\ h\ggg\to 0}}{\text{limit}}\left[\frac{CF'G'H'}{h\,k}\right]=\underset{\substack{k\ggg\to 0\\ h\ggg\to 0}}{\text{limit}}\left[\frac{\frac{h\,k}{\cos\beta}}{h\,k}\right]=\frac{1}{\cos\beta}.$$

100. Let ATL be any surface, and $ABCD - ON = V$, a volume limited by it, the three coördinate planes, and the planes DQR and BPS parallel, respectively, to XZ and YZ. From § 28, we have $V = f(x, y)$.



By the method used in the last Article, considering the corresponding volumes instead of the surfaces, we obtain.

$$\frac{d^2v}{dx\,dy} = \underset{\substack{h \ggg\!\!\to 0 \\ k \ggg\!\!\to 0}}{\text{limit}} \left[\frac{f(x+h,\, y+k) - f(x,\, y+k) - [f(x+h,\, y) - f(x,\, y)]}{h\,k} \right].$$

In which,

$$f(x+h, y+k) - f(x, y+k) - [f(x+h, y) - f(x, y)] = \text{vol.}\,CFGH - NM.$$

Hence, § 45, $\quad \dfrac{d^2v}{dx\,dy} = \underset{\substack{h \ggg\!\!\to 0 \\ k \ggg\!\!\to 0}}{\text{limit}} \left[\dfrac{CFGH - NM}{h\,k} \right] = NC = z;$

and $\quad \dfrac{d^2v}{dx\,dy}\,dx\,dy = z\,dx\,dy.$